International Mathematical Series • Volume 10

# SOBOLEV SPACES IN MATHEMATICS III

## Applications in Mathematical Physics

Victor Isakov
EDITOR

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## APPLICATIONS IN MATHEMATICAL PHYSICS

INTERNATIONAL MATHEMATICAL SERIES

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# SOBOLEV SPACES IN MATHEMATICS III

# Applications in Mathematical Physics

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Springer
Tamara Rozhkovskaya Publisher

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This series was founded in 2002 and is a joint publication of Springer and "Tamara Rozhkovskaya Publisher." Each volume presents contributions from the Volume Editors and Authors exclusively invited by the Series Editor Tamara Rozhkovskaya who also prepares the Camera Ready Manuscript. This volume is distributed by "Tamara Rozhkovskaya Publisher" (tamara@mathbooks.ru) in Russia and by Springer over all the world.

ISBN 978-0-387-85651-3 e-ISBN 978-0-387-85652-0
ISBN 978-5-901873-28-1 (Tamara Rozhkovskaya Publisher)
ISSN 1571-5485

Library of Congress Control Number: 2008937487



Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

springer.com

*To the memory of*
*Sergey L'vovich Sobolev*
*on the occasion of his centenary*

# Main Topics

Sobolev's discoveries of the 1930's have a strong influence on development of the theory of partial differential equations, analysis, mathematical physics, differential geometry, and other fields of mathematics. The three-volume collection *Sobolev Spaces in Mathematics* presents the latest results in the theory of Sobolev spaces and applications from leading experts in these areas.

*I. Sobolev Type Inequalities*
In 1938, exactly 70 years ago, the original Sobolev inequality (an embedding theorem) was published in the celebrated paper by S.L. Sobolev "On a theorem of functional analysis." By now, the Sobolev inequality and its numerous versions continue to attract attention of researchers because of the central role played by such inequalities in the theory of partial differential equations, mathematical physics, and many various areas of analysis and differential geometry. The volume presents the recent study of different Sobolev type inequalities, in particular, inequalities on manifolds, Carnot–Carathéodory spaces, and metric measure spaces, trace inequalities, inequalities with weights, the sharpness of constants in inequalities, embedding theorems in domains with irregular boundaries, the behavior of maximal functions in Sobolev spaces, etc. Some unfamiliar settings of Sobolev type inequalities (for example, on graphs) are also discussed. The volume opens with the survey article "My Love Affair with the Sobolev Inequality" by David R. Adams.

*II. Applications in Analysis and Partial Differential Equations*
Sobolev spaces become the established language of the theory of partial differential equations and analysis. Among a huge variety of problems where Sobolev spaces are used, the following important topics are in the focus of this volume: boundary value problems in domains with singularities, higher order partial differential equations, nonlinear evolution equations, local polynomial approximations, regularity for the Poisson equation in cones, harmonic functions, inequalities in Sobolev–Lorentz spaces, properties of function spaces in cellular domains, the spectrum of a Schrödinger operator with negative potential, the spectrum of boundary value problems in domains with cylindrical and quasicylindrical outlets to infinity, criteria for the complete integrability of systems of differential equations with applications to differential geometry, some aspects of differential forms on Riemannian manifolds related to the Sobolev inequality, a Brownian motion on a Cartan–Hadamard manifold, etc. Two short biographical articles with unique archive photos of S.L. Sobolev are also included.

*III. Applications in Mathematical Physics*

The mathematical works of S.L. Sobolev were strongly motivated by particular problems coming from applications. The approach and ideas of his famous book "Applications of Functional Analysis in Mathematical Physics" of 1950 turned out to be very influential and are widely used in the study of various problems of mathematical physics. The topics of this volume concern mathematical problems, mainly from control theory and inverse problems, describing various processes in physics and mechanics, in particular, the stochastic Ginzburg–Landau model with white noise simulating the phenomenon of superconductivity in materials under low temperatures, spectral asymptotics for the magnetic Schrödinger operator, the theory of boundary controllability for models of Kirchhoff plate and the Euler–Bernoulli plate with various physically meaningful boundary controls, asymptotics for boundary value problems in perforated domains and bodies with different type defects, the Finsler metric in connection with the study of wave propagation, the electric impedance tomography problem, the dynamical Lamé system with residual stress, etc.

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# Applications in Mathematical Physics

Victor Isakov Ed.

# Contents

# Preface

Victor Isakov

This volume contains various results on partial differential equations where Sobolev spaces are used. Their selection is motivated by the research interests of the editor and the geographical links to the places where S.L. Sobolev worked and lived: St. Petersburg, Moscow, and Novosibirsk. Most of the papers are written by leading experts in control theory and inverse problems. Another reason for the selection is a strong link to applied areas. In my opinion, control theory and inverse problems are main areas of differential equations of importance for some branches of contemporary science and engineering. S.L. Sobolev, as many great mathematicians, was very much motivated by applications. He did not distinguished between pure and applied mathematics, but, in his own words, between "good mathematics and bad mathematics." While he possessed a brilliant analytical technique, he most valued innovative ideas, solutions of deep conceptual problems, and not mathematical decorations, perfecting exposition, and "generalizations."

S.L. Sobolev himself never published papers on inverse problems or control theory, but he was very much aware of the state of art and he monitored research on inverse problems. In particular, in his lecture at a Conference on Differential Equations in 1954 (found in Sobolev's archive and made available to me by Alexander Bukhgeim), he outlined main inverse problems in geophysics: the inverse seismic problem, the electromagnetic prospecting, and the inverse problem of gravimetry. While at that time one of the main achievements was the solution of the one-dimensional Sturm–Liouville and spectral problems, he emphasized importance and possibilities of multi-dimensional inverse problems.

I was a part of Sobolev Seminar at the Institute of Mathematics, Novosibirsk, the USSR, from 1971 to 1980, until he left for Moscow. When presenting

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III*, *International Mathematical Series*. 
doi: 10.1007/978-0-387-85652-0, 

new results on inverse problems I was always aware that S.L. Sobolev followed presentations very closely and liked new ideas and not technicalities.

Now it is hard to imagine papers on the theory of partial differential equations written without use of Sobolev spaces. The papers in this collection are not an exception: almost all of them to some extent make use of these spaces. Their collection shows a variety of situations where Sobolev spaces are used. One of the main contributions of S.L. Sobolev is the introduction and use of Sobolev spaces and generalized (weak) solutions to boundary value problems to establish existence and uniqueness of generalized solutions to some basic boundary value problems (in particular, for hyperbolic and higher order elliptic equations) and their relations to classical solutions. Together with K.O. Friedrichs he can be viewed as a founding father of the contemporary theory of partial differential equations.

Now we briefly review the contents of the volume.

M. Belishev exposes an approach to the problem of reconstruction of a Riemannian $\Omega$ from the elliptic or hyperbolic Dirichlet-to-Neumann maps based on representation of algebras of functions. The basic idea is that these maps uniquely determine certain sufficiently large algebra of functions on $\Omega$, and hence the spectrum of this algebra which is isometric to $\Omega$. This approach applies to elliptic operators at least in the two-dimensional case, where there are strong connections between (generalized) analytic functions and elliptic partial differential equations. For second order hyperbolic equations the needed algebra is generated by the family of projection operators. A projection operator in this case projects onto the $L^2(\Omega)$-closure of waves sent from the boundary of $\Omega$ at times from 0 to $\theta < T$. This approach shows unusual aspects of the powerful and innovative Boundary Control Method initiated by M. Belishev and has a strong promise for new applications.

The memoir of A. Fursikov, M. Gunzburger, and J Peterson is devoted to the theory of boundary value problems for the stochastic Ginzburg–Landau equation. This semilinear equation models the famous phenomenon of superconductivity in materials under low temperatures. The authors introduce into the Ginzburg–Landau model (complex-valued) white noise. This change has some regularizing effect. In addition, the system enjoys some interesting ergodic properties. They demonstrate existence and uniqueness of weak solutions. The exposition contains a convenient definition and useful properties of the Wiener measures and of the Wiener process, discusses the Ito integral and the Ito formula. The existence of a weak solution is proved by obtaining a priori estimates of solutions, including the mean modulus of continuity, some special compactness theorems, and passing to limits when the size of spatial grid goes to zero. Finally, the existence of the strong statistical solution is established.

The paper of V. Isakov and N. Kim contains a first proof of Carleman type estimate with two large parameters for a general second order partial differential operator with real-valued coefficients. Carleman estimates are estimates in an $L^2$-space with the exponential weight containing a large parameter. They are the basic tool in proving uniqueness and stability of the continuation of solutions to partial differential equations. The continuation of solutions is of fundamental importance for boundary control theory and inverse problems. While their theory is well developed for scalar operators, systems of partial differential equations present currently a formidable challenge. Some classical isotropic systems can be principally diagonalized and handled like scalar equations, but more general and important anisotropic in some cases can be only transformed into principally triangular ones. Then two large parameters are essential to achieve global results. As an example, the classical Lamé system with residual stress is handled.

The paper of V. Ivrii, one of the best known students of Sobolev, discloses more detail about his deep research on spectral asymptotics. He considers the weighted integral of the spectral kernel of the magnetic Schrödinger operator. The (singular) weight is the inverse of the distance to the diagonal, so this integral can be interpreted as the Dirac correction term related to the ground state energy of atoms or molecules. The goal is to justify the Weyl type asymptotic behavior of this integral, i.e., to obtain best possible estimate for the remainder. The basic technique is the microlocal analysis and the leading idea is to make use of (micro)hyperbolicity. An essential tool of proofs is splitting integration domains into special zones, somehow in the spirit of the classical potential theory. Conditions for estimates of remainders include some assumptions on the distribution of eigenvalues and on microhyperbolicity of the Schrödinger operator.

In their contribution, I. Lasiecka and R. Triggiani develop a complete theory of boundary controllability for important partial differential equations modeling the Kirchhoff plate and the Euler–Bernoulli plate with various physically meaningful (linear and nonlinear) boundary controls. In addition, they consider the Schrödinger equation with the Dirichlet and Neumann boundary controls which is of significance for plates and shells equations due to their factorization into products of Schrödinger operators. One of the main goals is to achieve exact controllability (i.e., driving the system into zero final state by boundary control in a finite time) and the feedback stabilization of solutions for large times. The basic tool is the semigroup theory. Very delicate related questions concern continuity of boundary operators in Sobolev spaces. Several proofs are given and there are references to an extensive bibliography on the subject.

V. Maz'ya and A. Movchan obtain and justify asymptotic expansions of Green's functions of the transmission and Neumann problems for the Laplace equation in a domain with several holes. The important feature of their result is the uniformity of the asymptotic expansions with respect to the arguments of Green's functions. The small parameter $\varepsilon$ in these expansions is the average size of the holes. The authors employ the potential theory and the theory of

Sobolev spaces to obtain uniform bounds of the remainder in the asymptotic expansion of quadratic order with respect to $\varepsilon$.

M. Taylor is discussing the use of the Finsler metric to study the wave propagation. The Riemannian metric is a particular case of the Finsler metric and its importance for the theory of elliptic and hyperbolic equations is widely recognized. The author introduces the Finsler symbol, connects it to pseudodifferential operators and hyperbolic partial differential equations, and interprets a construction of the fundamental solution to the hyperbolic Cauchy problem by using the Finsler symbol. He gives an example of a strictly hyperbolic equation of the fourth order that gives rise to a non-Riemannian Finsler metric. Finally, he describes some applications to the ergodic theory and harmonic analysis. The main technique of this paper is the theory of pseudodifferential operators.

# Geometrization of Rings as a Method for Solving Inverse Problems

Mikhail Belishev

*To the memory of S.L. Sobolev*

**Abstract** In the boundary value inverse problems on manifolds, it is required to recover a Riemannian manifold $\Omega$ from its boundary inverse data (the elliptic or hyperbolic Dirichlet-to-Neumann map, spectral data, etc). We show that for a class of elliptic and hyperbolic problems the required manifold is identical with the spectrum of a certain algebra determined by the inverse data and, consequently, to recover the manifold it suffices to represent the corresponding algebra in the relevant canonical form.

## 1 Introduction

**1.1. About the paper.** "Rings" is the original name of what is now called "algebras" [10, 12], and "geometrization" is a representation of a commutative algebra in the form of a function algebra [1]. By the Gelfand theorem, any commutative Banach algebra (CBA) is canonically isomorphic to a subalgebra of the algebra $\mathcal{C}(\Omega)$ of continuous functions on a compact Hausdorff space $\Omega$. The role of $\Omega$ is played by the algebra spectrum (a properly topologized set of multiplicative functionals), whereas the canonical representation is realized by the Gelfand transform which maps elements of algebra to functions on $\Omega$. Thus, being an abstract notion defined by axioms, a CBA contains an intrinsic "geometric" object $\Omega$. The appearance of such an object from

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[1] In the case of an operator algebra, such a representation is also referred to as "diagonalization."

V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series.*

doi: 10.1007/978-0-387-85652-0, 

axioms looks like a miracle and, perhaps, is one of most beautiful phenomena in functional analysis.

A remarkable fact is that this phenomenon enables us to solve inverse problems. Namely, the problem of determination of a Riemannian manifold from its boundary data can be solved by the scheme

$$\text{“inverse data”} \Rightarrow \text{proper CBA} \Rightarrow \text{CBA's spectrum} \equiv \text{manifold}$$

The last implication is realized by the Gelfand transform. In [4], this scheme was applied for the first time to the 2D Calderon problem consisting in the determination of a Riemann surface from its *elliptic* Dirichlet-to-Neumann map. In [4], we also conjectured that the procedure recovering a manifold from its *hyperbolic* Dirichlet-to-Neumann map (the boundary control method: see [1, 3]) can be also interpreted as a version of the Gelfand transform. In this paper, we justify this conjecture and provide such an interpretation.

What we propose is not an universal method for solving inverse problems (such a method does not exist!), but a unified "view from a height" at a class of very different problems. Our considerations are essentially based on facts and ideas of functional analysis. Fruitful applications of functional analysis in mathematical physics were perfectly demonstrated by the outstanding mathematician S.L. Sobolev in his celebrated book [13]. We dedicate this paper to his memory.

**1.2. Content.** In Sect. 1, we shortly recall some basic facts about CBAs and $C^*$-algebras and then specify this information for elliptic and hyperbolic problems in Subsects. 2.1 and 3.1 respectively. In Sect. 2, we describe a renewed version of the approach [4] to the 2D elliptic (Calderon) problem. As is shown, to recover a Riemann surface from its Dirichlet-to-Neumann (DN) map is to determine the crown of a certain function algebra, determined by the DN map, on the boundary. In Sect. 3, a *hyperbolic dynamical system with boundary control* is introduced. Such a system can be realized in the canonical form so that the realization possesses the features of "intuitive hyperbolicity:" its states propagate into a compact set with finite speed. To construct the realization is to diagonalize a certain operator algebra determined by relevant "boundary inverse data." The problem of determination of a Riemannian manifold from its response operator or spectral data fits this scheme and is considered as an example.

We use the following abbreviations:

CBA — commutative Banach algebra

cHs — compact Hausdorff space

DSBC — dynamical system with boundary control

$G$-transform — Gelfand transform

IP — inverse problem

DN — Dirichlet-to-Neumann

Throughout the paper, “smooth” means “$C^\infty$-smooth.” All Hilbert spaces are separable. The identity operator is denoted by $\mathbb{I}$.

## 2 Algebra Handbook

**2.1. CBAs, G-transform.** We recall some facts about commutative Banach algebras (see, for example, [9, 10] for details) and introduce the Gelfand transform which is the main device for solving IPs.

(1) A CBA is a (complex or real) Banach space $\mathcal{A}$ equipped with the multiplication operation $ab$ satisfying $ab = ba$, $\|ab\| \leqslant \|a\|\,\|b\|$, $a, b \in \mathcal{A}$. Until the otherwise is not specified, we consider algebras with the unit $e \in \mathcal{A}$, $ea = ae = a$. *Example*: The algebra $\mathcal{C}(X)$ of continuous functions on a cHs $X$ with the norm $\|a\| = \max_X |a(\cdot)|$. Subalgebras of $\mathcal{C}(X)$ are called *function algebras*. A CBA is said to be *uniform* if $\|a^2\| = \|a\|^2$. All function algebras are uniform.

(2) Let $\mathcal{A}'$ be the space of linear continuous functionals on $\mathcal{A}$. A functional $\delta \in \mathcal{A}'$ is called *multiplicative* if $\delta(ab) = \delta(a)\delta(b)$. *Example*: A Dirac measure $\delta_{x_0} \in \mathcal{C}'(X)$: $\delta_{x_0}(a) = a(x_0)$. Each multiplicative functional is of the norm 1. The set of multiplicative functionals endowed with the $*$-week topology (in $\mathcal{A}'$) is called the *spectrum* of $\mathcal{A}$ and is denoted by $\Omega_{\mathcal{A}}$. The spectrum is a cHs.

(3) The *G–transform* acts from a CBA $\mathcal{A}$ into $\mathcal{C}(\Omega_{\mathcal{A}})$ by the rule $G : a \mapsto a(\cdot)$, $a(\delta) := \delta(a)$, $\delta \in \Omega_{\mathcal{A}}$ and, consequently, maps $\mathcal{A}$ to a function algebra. The passage from $\mathcal{A}$ to $G\mathcal{A} \subset \mathcal{C}(\Omega_{\mathcal{A}})$ is referred to as *geometrization.*

**Theorem 2.1 (**Gelfand). *If $\mathcal{A}$ is a uniform CBA, then $G$ is an isometric isomorphism from $\mathcal{A}$ onto $G\mathcal{A}$, i.e., $G(\alpha a + \beta b + cd) = \alpha Ga + \beta Gb + Gc\,Gd$ and $\|Ga\| = \|a\|$ for all $a, b, c, d \in \mathcal{A}$ and numbers $\alpha, \beta$.*

In what follows, we deal only with uniform CBAs.

(4) If two CBAs $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic (we write $\mathcal{A} \equiv \mathcal{B}$) through an isometry $j$, then $G\mathcal{A} \equiv G\mathcal{B}$, whereas the conjugate isometry $j^* : \mathcal{B}' \to \mathcal{A}'$ is a homeomorphism between their spectra: $j^*\Omega_{\mathcal{B}} = \Omega_{\mathcal{A}}$ (so, denoting a homeomorphism by $\asymp$, we have $\Omega_{\mathcal{A}} \asymp \Omega_{\mathcal{B}}$).

(5) Let $\mathcal{A}(X) \subset \mathcal{C}(X)$ be a closed function algebra. For each $x_0 \in X$ the Dirac measure $\delta_{x_0}$ belongs to $\Omega_{\mathcal{A}(X)}$. Therefore, identifying $x_0 \equiv \delta_{x_0}$, we get the canonical embedding $X \subset \Omega_{\mathcal{A}(X)}$.

(6) In general, the passage from $X$ to $\Omega_{\mathcal{A}(X)}$ can preserve or extend $X$.[2] In the last case, the set $\Omega_{\mathcal{A}(X)} \setminus X \neq \varnothing$ is called a *crown* of $\mathcal{A}(X)$, whereas the G-transform extends functions of $\mathcal{A}(X)$ from $X$ to the crown.

(7) An algebra $\mathcal{A}(X)$ is said to be *generic* if the Dirac measures exhaust its spectrum: $\Omega_{\mathcal{A}(X)} = X$. In other words, it is an algebra without crown. A generic algebra is identical to its G-transform: $G\mathcal{A}(X) \equiv \mathcal{A}(X)$. On the other hand, any $G\mathcal{A}$ (as a subalgebra of $\mathcal{C}(\Omega_{\mathcal{A}})$) is automatically generic, i.e., $GG = G$ holds and implies $\Omega_{G\mathcal{A}} = \Omega_{\mathcal{A}}$.

**2.2. $C^*$-algebras.** For a subset $\mathcal{S}$ of an algebra we denote by $\bigvee \mathcal{S}$ the (minimal) subalgebra generated by $\mathcal{S}$. In the case of an operator algebra, u-clos$\bigvee\mathcal{S}$, s-clos$\bigvee\mathcal{S}$, and w-clos$\bigvee\mathcal{S}$ denote the closures of this subalgebra in the uniform (norm), strong, and week operator topologies respectively.

(8) A $C^*$-*algebra* is a Banach algebra endowed with an *involution* $(^*)$ satisfying $(\alpha a + \beta b + cd)^* = \overline{\alpha} a^* + \overline{\beta} b^* + d^* c^*$ and $\|a^* a\| = \|a\|^2$. We assume that a $C^*$-algebra contains the unit. *Example*: The algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators in a Hilbert space $\mathcal{H}$ with the operator norm and conjugation.

**Theorem 2.2** (Gelfand–Naimark). *Any commutative $C^*$-algebra $\mathcal{A}$ is isometrically isomorphic* (*through the* G-*transform*) *to $\mathcal{C}(\Omega_{\mathcal{A}})$, and $G(a^*) = \overline{Ga}$.*

(9) An operator algebra $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ is said to be *cyclic* if there is a (*cyclic*) element $h \in \mathcal{H}$ such that the set $\{Ah|\ A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

**2.3. Neumann algebras.**

(10) A $w$-closed $C^*$-algebra $\mathcal{N} \subset \mathfrak{B}(\mathcal{H})$ is called a *Neumann algebra. Example*: Let $X$ be a cHs endowed with a finite Borel measure $\mu$, $\mathcal{H} = L_{2,\,\mu}(X)$. For $m \in L_{\infty,\,\mu}(X)$ we denote by ad $m$ the operator multiplying functions by $m$. The algebra of bounded multiplicators $\mathcal{N} = \text{ad}\ L_{\infty,\,\mu}(X)$ is a commutative cyclic Neumann algebra in $\mathcal{H}$[3]. The following result shows that this example is of universal character.

**Theorem 2.3** (Gelfand–Naimark–Segal). 1. *Any $C^*$-algebra is isometrically isomorphic to a subalgebra of $\mathfrak{B}(\mathcal{H})$.*

2. *Let $\mathcal{N}$ be a commutative cyclic Neumann algebra in $\mathcal{H}$, $\mathbb{I} \in \mathcal{N}$. There exists a cHs $X$, a measure $\mu$ on $X$, and a unitary operator $U : \mathcal{H} \to L_{2,\,\mu}(X)$ such that $U\mathcal{N}U^* = \text{ad}\ L_{\infty,\,\mu}(X)$.*

Such a canonical representation of $\mathcal{N}$ is referred to as a *diagonalization*, which is a synonym of "geometrization" for operator algebras.

[2] See examples in [10, Chapt. III, Part 11, no 3].

[3] Any $h \in L_{2,\,\mu}(X)$ satisfying $h(\cdot) \neq 0$ a.e. can serve as a cyclic element.

If $\mathcal{H}$ is a real Hilbert space and $\mathcal{N}$ is commutative, cyclic, and consisting of the self-adjoint operators, then $\mathcal{N}$ is isometrically isomorphic to the algebra of the bounded multiplicators in a real $L_{2,\,\mu}(X)$.

(11) A Neumann algebra is determined by its projections: $\mathcal{N} = \text{s-clos}\bigvee\mathcal{P}$, where $\mathcal{P} := \{P \in \mathcal{N} \mid P^* = P,\ P^2 = P\}$.

# 3 Elliptic Inverse Problem

**3.1. Shilov's boundary.** We provide some additional facts about CBAs which will be used in the treatment of elliptic IP.

(12) Let $\mathcal{A}$ be a CBA. A set $B \subset \Omega_{\mathcal{A}}$ is called a *boundary* if for every $a(\cdot) \in G\mathcal{A}$ the modulo $|a(\cdot)|$ attains the maximum on $B$. Let $\beta[\mathcal{A}]$ be the set of boundaries, The minimal boundary $\partial\mathcal{A} := \bigcap\limits_{B\in\beta[\mathcal{A}]} B$ does exist and is called *Shilov's boundary* of $\mathcal{A}$. *Shilov's boundary* is a compact subset in $\Omega_{\mathcal{A}}$. For generic $\mathcal{A}(X)$ we consider $\partial\mathcal{A}(X)$ as a subset of $X$.

(13) Introduce a *trace map* $\text{tr} : a(\cdot) \mapsto a(\cdot)|_{\partial\mathcal{A}}$. By definitions, tr preserves the algebraic operations and sup-norm. In other words, tr is an isometric isomorphism from $G\mathcal{A}$ onto its image $\text{tr}\, G\mathcal{A}$ which is a closed subalgebra of $\mathcal{C}(\partial\mathcal{A})$. Hence $\text{tr}\, G\mathcal{A} \equiv G\mathcal{A}$, which yields $\Omega_{\text{tr}G\mathcal{A}} \asymp \Omega_{G\mathcal{A}} \asymp \Omega_{\mathcal{A}}$ (see (7)). Moreover, if $\mathcal{A}(X)$ is generic (i.e., $\Omega_{\mathcal{A}(X)} \asymp X$), then

$$\Omega_{\text{tr}G\mathcal{A}(X)} \asymp X\,. \tag{3.1}$$

(14) Let $\mathcal{L}$ be a linear space, and let $\mathfrak{r}$ be a subspace of $\mathcal{L}\times\mathcal{L}$. For $a, b \in \mathcal{L}$ we write $a\,\mathfrak{r}\,b$ if $\{a, b\} \in \mathfrak{r}$ and call $\mathfrak{r}$ a *linear relation.*

Any complex CBA $\mathcal{A}$ determines the relation

$$\star := \{\{\Re a(\cdot), \Im a(\cdot)\} |\ a(\cdot) \in G\mathcal{A}\} \subset \mathcal{C}_{\text{real}}(\Omega_{\mathcal{A}}) \times \mathcal{C}_{\text{real}}(\Omega_{\mathcal{A}}).$$

Therefore, the relation

$$\begin{aligned}\mathfrak{h} := \{\{\Re y, \Im y\} |\ y \in \text{tr}\, G\mathcal{A}\} &= \{\{\Re\,\text{tr}\, a(\cdot), \Im\,\text{tr}\, a(\cdot)\} |\ a(\cdot) \in G\mathcal{A}\} \\ \subset \mathcal{C}_{\text{real}}(\partial\mathcal{A}) \times \mathcal{C}_{\text{real}}(\partial\mathcal{A})\end{aligned}$$

is well defined and determines the boundary algebra $\text{tr}\, G\mathcal{A}$ as follows: $y = f + f_*\text{i} \in \text{tr}\, G\mathcal{A}$ if $f\mathfrak{h}f_*$. With a slight abuse of notation, we can write

$$\mathfrak{h} \;=\; \text{tr}\,\star\,. \tag{3.2}$$

(15) Suppose that $\Omega$ is a cHs, $\mathcal{A}(\Omega)$ is a generic algebra, $\Gamma := \partial\mathcal{A}(\Omega) \subset \Omega$ is its Shilov's boundary, $\mathcal{A}(\Gamma) := \operatorname{tr} G\mathcal{A}(\Omega)$, and $\mathfrak{h} \in \mathcal{C}_{\rm real}(\Gamma) \times \mathcal{C}_{\rm real}(\Gamma)$. Assume that $\Gamma$ and $\mathfrak{h}$ are given. Then we can recover $\mathcal{A}(\Omega)$ (up to an isometric isomorphism) by the following scheme (see also Fig. 1):

*Step* 1. Determine $\mathcal{A}(\Gamma) = \{y \in \mathcal{C}(\Gamma)|\ \Re y\, \mathfrak{h}\, \Im y\}$.

*Step* 2. Find $\Omega_{\mathcal{A}(\Gamma)} \asymp \langle$see (3.1)$\rangle \asymp \Omega$. Identifying $\Gamma \ni \gamma \equiv \delta_\gamma \in \Omega_{\mathcal{A}(\Gamma)}$, attach $\Omega$ to $\Gamma$.

*Step* 3. Determine $G\mathcal{A}(\Gamma) \equiv \mathcal{A}(\Omega)$[4].

The bonus for so long and abstract introduction is that now we can solve the 2D Calderon problem just by applying this scheme.

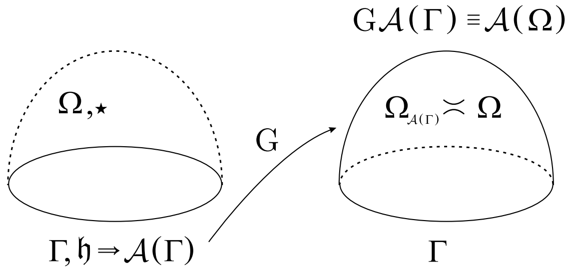


**Fig. 1** Recovering $\mathcal{A}(\Omega)$.

**3.2.** $2D$ **Calderon problem.** Suppose that $\Omega$ is a 2-dimensional smooth compact orientable Riemannian manifold (surface) with boundary $\Gamma := \partial\Omega$, $g$ is a metric tensor on $\Omega$, $\Delta$ is the Beltrami–Laplace operator, $\nu = \nu(\gamma)$, $\gamma \in \Gamma$, is the outward normal. We assume that $\Omega$ is oriented and denote by $\mu$ the volume form; $\mu_\Gamma := \mu(\nu, \cdot)$ is the induced form at the boundary. Let $d$ be the exterior derivative on forms, $\star$ the Hodge operator, and $\delta$ the codifferential.

We consider the elliptic boundary value problem

$$\Delta u = 0 \qquad \text{in} \quad \Omega \setminus \Gamma, \tag{3.3}$$

$$u = f \qquad \text{on} \quad \Gamma, \tag{3.4}$$

with a real-valued function $f \in L_2(\Gamma)$. Let $u = u^f(x)$ be a solution. With this problem it is possible to associate the DN map $\Lambda : L_2(\Gamma) \to L_2(\Gamma)$, $\operatorname{Dom} \Lambda = \mathcal{C}^\infty(\Gamma)$, $\Lambda f := \nu \cdot \nabla u^f = \frac{\partial u^f}{\partial \nu}$.

[4] In fact, realizing this G-transform, we continue the functions $y \in \mathcal{A}(\Gamma)$ from $\Gamma$ onto the crown $\Omega \setminus \Gamma$ of the algebra $\mathcal{A}(\Gamma)$.

The Calderon problem is to determine $(\Omega, g)$ from a given $\Lambda$. In applications, it is also known as the *electric impedance tomography* problem: determine the shape of a conducting shell from the measurements at its border, $u^f$ being interpreted as an electric potential (voltage). The problem is solved (see [8, 4]). Here, we describe a renewed version of the geometrization approach [4].

**3.3. Algebras $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Gamma)$.** One of the basic objects of Riemann surface theory is the algebra of analytic functions $\mathcal{A}(\Omega) := \{w = u + u_*\mathrm{i} |\ u, u_* \in \mathcal{C}_{\rm real}(\Omega);\ du_* = \star\, du \text{ in } \Omega \setminus \Gamma\}$ ([6]; see also [4]). It has the following properties.

- A linear relation $u \star u_*$ determining $\mathcal{A}(\Omega)$ is the Cauchy–Riemann condition $du_* = \star\, du$.
- $\mathcal{A}(\Omega)$ is a generic algebra: $\Omega_{\mathcal{A}(\Omega)} = \Omega,\ G\mathcal{A}(\Omega) \equiv \mathcal{A}(\Omega)$ .
- By the maximum principle for analytic functions, Shilov's boundary coincides with the topological boundary: $\partial\mathcal{A}(\Omega) = \partial\Omega = \Gamma$. Hence the algebra $\mathcal{A}(\Gamma) := \mathrm{tr}\mathcal{A}(\Omega) = \{w|_\Gamma \mid w \in \mathcal{A}(\Omega)\}$ is isometrically isomorphic to $\mathcal{A}(\Omega)$.

As a result, if $\Gamma$ and the relation $\mathfrak{h}$ determining $\mathcal{A}(\Gamma)$ are given, then one can recover $\Omega$ (up to a homeomorphism) and $\mathcal{A}(\Omega)$ (up to an isometric isomorphism) by the scheme (15) in Subsect. 3.1.

**3.4. Hilbert transform.** We introduce an intrinsic operator associated with the Calderon problem.

Let $\theta$ be a tangent vector field on $\Gamma$, $|\theta| = 1, \mu(\theta, -\nu) = 1$. Understanding $\theta$ as a differentiation, for a smooth function $f = f(\gamma)$ on $\Gamma$ we denote by $df := \theta f$ the corresponding derivative.

Let $\dot{L}_2(\Gamma) := L_2(\Gamma) \ominus \{\text{constants}\}$ be the subspace of zero mean value functions. Denote by $J : \dot{L}_2(\Gamma) \to \dot{L}_2(\Gamma)$ the integration

$$d\,Jf = f\,, \qquad \int\limits_\Gamma Jf\ \mu_\Gamma = 0\,, \qquad J^* = -J\,.$$

An operator $H : \dot{L}_2(\Gamma) \to \dot{L}_2(\Gamma)$, $H := \Lambda J$, is called the *Hilbert transform.* By elliptic theory, $H$ is a well defined continuous operator.

A key point is that the Hilbert transform determines the trace algebra. Namely, as is shown in [4], the following assertion holds.

**Lemma 3.1.** *A smooth complex function $y = f + f_*\mathrm{i}$ belongs to $\mathcal{A}(\Gamma)$ (i.e., $y = w|_\Gamma$ for a $w \in \mathcal{A}(\Omega)$) if and only if $df, df_* \in \mathrm{Ker}(\mathbb{I} + H^2)$ and $df_* = H df$.*

In other words, $H$ determines $\mathfrak{h}$ which, in turn, does determine $\mathcal{A}(\Gamma)$.

**3.5. Solving the problem.** If $\Lambda$ is given, one can recover $\Omega$ by repeating the procedure (15) in Subsect. 3.1. Namely:

*Step* 1. Find $H = \Lambda J$, determine the relation $\mathfrak{h}$, and recover the trace algebra $\mathcal{A}(\Gamma)$.

*Step* 2. Find $\Omega_{\mathcal{A}(\Gamma)} \asymp \langle$ see (4.1) $\rangle \asymp \Omega$[5]. Identifying $\Gamma \ni \gamma \equiv \delta_\gamma \in \Omega_{\mathcal{A}(\Gamma)}$, attach $\Omega$ to $\Gamma$. Thus, the shell is determined up to a homeomorphism.

*Step* 3. Applying the G-transform, get $G\mathcal{A}(\Gamma) \equiv \mathcal{A}(\Omega)$. Thus, analytic functions on the shell are also recovered.

So, the shell is determined as the crown of the trace algebra. Extra efforts are required to endow it with $2D$-differentiable structure and Riemannian metric¿ For this purpose, one can use $\{\Re w, \Im w\}$, $w \in \mathcal{A}(\Omega)$ as local coordinates on $\Omega$ (see [4] for details). Note that the metric can be determined not uniquely, but up to a conformal deformation.

Another problem is to characterize the inverse data, i.e., to find necessary and sufficient conditions for $\Lambda$ to be a DN-map of a shell. An efficient analytic characterization is proposed in [7]. Note that the description of $\mathfrak{h}$ in terms of operator $H$, which is given by Lemma 3.1, provides the sharp necessary conditions of the algebraic nature on the Hilbert transform: the set $\{f + f_*\mathrm{i} \mid f_*\mathfrak{h}f\}$ must be a proper subalgebra in $\mathcal{C}(\Gamma)$.

**3.6. System $\alpha$. Diagonalization.** Here, we present one more possible look at the determination $\Lambda \Rightarrow \Omega$. We consider the problem (3.3), (3.4) as a system $\alpha$ and equip it with the standard system theory attributes.

With a solution $u^f$ we associate a 1-form (*state* or *electric field*) $e^f := du^f \in \mathcal{H}$, where $\mathcal{H} := \{e = d\psi \mid \psi \in H^1(\Omega), \delta e = 0\}$ is the space of exact harmonic fields (see, for example, [11]) regarded as a subspace of

$$\oplus \int_\Omega T^*_x(\Omega)\ \mu_x$$

and endowed with the corresponding $L_2$-metric

$$(e', e'')_{\mathcal{H}} := \int_\Omega e' \cdot e''\ \mu\,.$$

The space $\mathcal{H}$ plays a role of an *inner space* of the system $\alpha$. Note that the Hodge star $\star$ is a unitary operator in $\mathcal{H}$ acting pointwise: $(\star\, e)(x) = \star_x\, e(x)$ in $T^*_x(\Omega)$, $x \in \Omega$.

An *outer space* is $\mathcal{F} := \mathrm{Ker}\ (\mathbb{I} + H^2) \subset \dot{L}_2(\Gamma)$. An operator $W : df \mapsto e^f$ realizes the "input $\to$ state" correspondence; it is a continuous map from $\mathcal{F}$ to $\mathcal{H}$. It is easy to see that the operator equality

$$\star\, W = WH \tag{3.5}$$

is just a relevant version of (3.2).

[5] This is a key point: a cHs $\Omega$ appears as a result of the geometrization!

Integrating by parts, we find

$$
\begin{aligned}
(e^f, e^g)_{\mathcal{H}} = (Wf, Wg)_{\mathcal{H}} &= \int_{\Omega} du^f \cdot du^g \ \mu \\
= \int_{\Gamma} (\Lambda f)\, g \ \mu_\Gamma &= \int_{\Gamma} (Cdf)\, dg \ \mu_\Gamma = (Cdf, dg)_{\mathcal{F}}\,,
\end{aligned} \tag{3.6}
$$

where $C := W^*W = J^*\Lambda J = -JH$ is the so-called *connecting operator* of the system $\alpha$ acting in its outer space.

The analytic function algebra $\mathcal{A}(\Omega)$ determines an operator algebra in the inner space $\mathcal{H}$ as follows. In $T^*_x(\Omega)$, choose an orthonormal basis $e_1(x), e_2(x)$ : $\mu\,(e_1(x), e_2(x)) = 1$ and represent $e \in \mathcal{H}$ as $e(x) = c_1(x)e_1(x) + c_2(x)e_2(x)$, $x \in \Omega$. For $w = a + a_*\mathrm{i} \in \mathcal{A}(\Omega)$ we define[6] $\mathrm{ad}\, w : \mathcal{H} \to \mathcal{H}, (\mathrm{ad}\, w)e := (ac_1 - a_*c_2)e_1 + (a_*c_1 + ac_2)e_2$. Let $\mathrm{ad}\, \mathcal{A}(\Omega) \subset \mathfrak{B}(\mathcal{H})$ be the subalgebra of such operators. It is easy to check that $\mathrm{ad}\, \mathcal{A}(\Omega)$ is a well-defined object, and the correspondence $w \mapsto \mathrm{ad}\, w$ is an isometric isomorphism between the CBAs $\mathcal{A}(\Omega)$ and $\mathrm{ad}\, \mathcal{A}(\Omega)$.

In the outer space $\mathcal{F}$, every $\mathrm{tr}w = a|_\Gamma + a_*|_\Gamma\mathrm{i}$ determines an operator $\mathrm{ad}_\Gamma w : df \mapsto (a|_\Gamma \mathbb{I} + a_*|_\Gamma H)df$. It is easy to check that this operator is well defined. Let $\mathrm{ad}_\Gamma \mathcal{A}(\Omega) \subset \mathfrak{B}(\mathcal{F})$ be the subalgebra of such operators. The relation

$$
(\mathrm{ad}\, w)\, W \;=\; W\, \mathrm{ad}_\Gamma w, \qquad w \in \mathcal{A}(\Omega) \tag{3.7}
$$

follows from definitions, and (3.5) is its particular case for $a = 0$, $a_* = 1$.

Assume that $\mathcal{F}$ is endowed with a new Hilbert metric $(df, dg)_{\widetilde{\mathcal{F}}} := (Cdf, dg)_{\mathcal{F}}$ determined by the connecting operator. Then, by (3.6), $W : \widetilde{\mathcal{F}} \to \mathcal{H}$ is an isometry. Taking into account $(\mathbb{I} + H^2)df = 0$, we have

$$
\begin{aligned}
(Hdf, Hdg)_{\widetilde{\mathcal{F}}} &= (-JH\, Hdf, Hdg)_{\mathcal{F}} = (df, -JHdg)_{\mathcal{F}} \\
&= (-JHdf, dg)_{\mathcal{F}} = (df, dg)_{\widetilde{\mathcal{F}}}\,.
\end{aligned}
$$

Hence $H$ is a unitary operator in $\widetilde{\mathcal{F}}$ (as $\star$ is in $\mathcal{H}$), whereas (3.5) shows that the operator $W$ can be regarded as a *transform that diagonalizes* the Hilbert transform $H$. In the mean time, $W$ is, in fact, identical to the G-transform of the algebra $\mathrm{ad}_\Gamma \mathcal{A}(\Omega) \ni H$ determined by the inverse data.

**3.7. Comments.** So, to solve the Calderon problem, we construct a transform $W$ which diagonalizes the Hilbert transform and transfers it to the Hodge operator. We stress this fact because all these objects do have the natural multidimensional analogues (see [5]) and the relevant $W$ makes the same: it transfers $H$ to $\star$. However, the Calderon problem in dimension $n \geqslant 3$

---

[6] With a slight abuse of the notation: $\mathrm{ad}\, w$ multiplies by not a function, but a matrix $\begin{pmatrix} a & -a_* \\ a_* & a \end{pmatrix}$.

is so far open: roughly speaking, it is not clear, where $\Omega$ can be taken from. The point is that, in the $2D$ case, $\Omega$ appears as the spectrum of a CBA, i.e., as a result of diagonalizing not a single operator $H$, but an algebra $\mathrm{ad}_\Gamma \mathcal{A}(\Omega) \ni H$ (see (3.7)). Unfortunately, in the case of higher dimensions, such a "surrounding algebra" is not found yet (if it exists).

# 4 Hyperbolic Inverse Problem

**4.1. More about Theorem 2.3.** It is important to note that the cHs $X$ in the representation $\mathcal{N} \equiv L_{\infty,\,\mu}(X)$ is not determined by $\mathcal{N}$ up to a homeomorphism. Analyzing the proof (see, for example, [10, Theorem 4.4.3]), we see that $X$ appears as the spectrum of an $u$-closed subalgebra $\mathcal{A} \subset \mathcal{N}$, $\mathcal{A}^* = \mathcal{A}$ provided that s-clos$\mathcal{A} = \mathcal{N}$. Any $\mathcal{A}$ possessing these properties is available, and it can be chosen in an arbitrary way. Once a choice has been made, we apply the G-transform to $\mathcal{A}$ (in particular, find $\Omega_{\mathcal{A}} =: X$), get $G\mathcal{A} = \mathcal{C}(X)$ (see Theorem 2.2), and only then realize $\mathcal{N}$ as ad $L_{\infty,\,\mu}(X)$ in $L_{2,\,\mu}(X)$, whereas $\mathcal{A}$ is transferred onto ad $\mathcal{C}(X) \subset$ ad $L_{\infty,\,\mu}(X)$. In other words, one diagonalizes not $\mathcal{N}$, but a pair $\{\mathcal{N}, \mathcal{A}\}$ by choosing $\mathcal{A}$, which will play the role of continuous functions. We say that $\mathcal{A}$ is a *supporting algebra* for $\mathcal{N}$. In the problem under consideration, the choice of $\mathcal{A}$ is well motivated.

A measure $\mu$ appears in the diagonalization process as follows. Assume that a supporting algebra $\mathcal{A} \subset \mathcal{N}$ is chosen. Let $G : \mathcal{A} \to \mathcal{C}(X)$ be its G-transform. Choose a cyclic element $h \in \mathcal{H}$[7]. For $\varphi \in \mathcal{C}(X)$ we define the *integral* (i.e., $I \in \mathcal{C}'(X)$ and $\varphi \geqslant 0$ implies $I(\varphi) \geqslant 0$) by the formula $I(\varphi) := \big([G^{-1}\varphi]h, h\big)_{[\mathcal{H}]}$. By the Riesz–Kakutani theorem, there exists a unique measure $\mu$ on $X$ such that

$$I(\varphi) = \int_X \varphi \, d\mu.$$

Then

$$\begin{aligned}\|\varphi\|^2_{L_{2,\,\mu}(X)} &= \int_X |\varphi|^2 d\mu = \big(\big[G^{-1}(\overline{\varphi}\varphi)\big]\, h, h\big)_{\mathcal{H}} \\ &= \big([G^{-1}\varphi]^*[G^{-1}\varphi]h, h\big)_{\mathcal{H}} = \|[G^{-1}\varphi]h\|^2_{\mathcal{H}}\,.\end{aligned}$$

Hence the map $U : \mathcal{H} \to L_{2,\,\mu}(X)$, Dom $U = \{ah|\ a \in \mathcal{A}\}$, $U : ah \mapsto Ga$ is a densely defined isometry and can be extended to a unitary operator from $\mathcal{H}$ onto $L_{2,\,\mu}(X)$, whereas $U\mathcal{A}U^* =$ ad $\mathcal{C}(X) \subset$ ad $L_{\infty,\,\mu}(X)$ and $U\mathcal{N}U^* =$ ad $L_{\infty,\,\mu}(X)$. It is the transform $U$ which diagonalizes $\mathcal{N}$.

[7] Since $\mathcal{A}$ is $s$-dense in $\mathcal{N}$, $h$ is also a cyclic element of $\mathcal{A}$.

**4.2. System $\alpha^T$.** All Hilbert spaces and algebras used in the rest of the paper are assumed to be real. The object introduced below can be specified as a version of an abstract dynamic system with boundary control (DSBC) introduced in [2].

Let $\Gamma$ be a set with measure $\lambda$. Denote $\mathcal{G} := L_{2,\,\lambda}(\Gamma)$ and $\mathcal{F}^T := L_2\left([0,T];\mathcal{G}\right)$. We say that $\Gamma$ is a *controlling set*, $t \in [0,T]$ is a time ($T > 0$ is a final moment), $\mathcal{F}^T$ is an *outer space*, its elements are called *controls*.

Let $\mathcal{H}$ be a Hilbert space called an *inner space*, $W : \mathcal{F}^T \to \mathcal{C}\left([0,T];\mathcal{H}\right)$ a linear continuous map; the images $u^f := Wf$ are called *trajectories*. A "control→state" map (*control operator*) $W^T : \mathcal{F}^T \to \mathcal{H}$, $W^T f := u^f(T)$ is also well defined. The operator $C^T := (W^T)^* W^T$ acting in the outer space $\mathcal{F}^T$, is called a *connecting operator*.

In addition, we assume that $W$ is *causal* and possesses the *steady state property* which means the following. Let $D^{T,\,\xi}$ be a *delay operator* acting on functions of time $y = y(t)$ by the rule

$$(D^{T,\,\xi}y)(t) := \begin{cases} 0, & t \in [0, T-\xi), \\ y\left(t - (T-\xi)\right), & t \in [T-\xi, T] \end{cases}$$

($\xi$ is a parameter, $0 \leqslant \xi \leqslant T$). The above-mentioned property means the relation $WD^{T,\,\xi} = D^{T,\,\xi}W$. It implies $u^f(\xi) = W^T D^{T,\,\xi} f$ and hence, to know $W^T$ is to know $W$.

All the above-introduced objects are determined by the set $\{\Gamma, \lambda; \mathcal{H}; W^T\}$ which will be referred to as the DSBC $\alpha^T$.

The example, which inspires these abstract definitions, is the following. Let $\Omega$ be an $n$-dimensional smooth compact Riemannian manifold[8], $\Gamma := \partial\Omega$ the boundary, $\Delta$ the Beltrami–Laplace operator in $\Omega$. The system $\alpha^T_{\rm Riem}$ is associated with the boundary value initial problem

$$u_{tt} - \Delta u = 0 \qquad \text{in } (\Omega \setminus \Gamma) \times (0,T), \tag{4.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \Omega, \tag{4.2}$$

$$\nu \cdot \nabla u = f \qquad \text{on } \Gamma \times [0,T], \tag{4.3}$$

where $\nu$ is the outward normal, $u = u^f(x,t)$ is the solution. Here, the inner space is $\mathcal{H} = L_2(\Omega)$ (with measure vol), $u^f(\cdot,t) \in \mathcal{H}$ is a time dependent state (*wave*), $u^f \in \mathcal{C}\left([0,T];\mathcal{H}\right)$ is a trajectory initiated by a Neumann boundary control $f \in \mathcal{F}^T = L_2\left([0,T];L_2(\Gamma)\right)$, and $W^T f := u^f(\cdot,T)$. The system $\alpha^T_{\rm Riem}$ is determined by the collection $\{\Gamma, \mathrm{vol}_\Gamma; L_2(\Omega); W^T\}$. Regarding the final moment, we assume that $T > \mathrm{diam}\,\Omega$.

**4.3. Algebras $\mathcal{N}$ and $\mathcal{T}$.** We return to a general DSBC $\alpha^T$ and recall that a control $f \in \mathcal{F}^T$ is a $\mathcal{G}$-valued function of time: for each $t$, $f(t)$ is an element of $L_{2,\,\lambda}(\Gamma)$.

---

[8] The orientability of $\Omega$ is not necessary

Denote $\Sigma_\Gamma^+ := \{\sigma \subset \Gamma |\ \lambda(\sigma) > 0\}$. With each $\sigma \in \Sigma_\Gamma^+$ we associate the family of subspaces

$$\mathcal{F}_\sigma^{T,\,\xi} := \left\{f \in \mathcal{F}^T \,\middle|\, f|_{[0,T-\xi]} = 0,\ \operatorname{supp} f(t) \subset \sigma\right\}$$

parametrized with $\xi \in [0,T]$ and formed by the delayed controls[9] supported in $\sigma$. The set of the corresponding states

$$\mathcal{U}_\sigma^\xi := W^T \mathcal{F}_\sigma^{T,\,\xi} = \left\{u^f(\xi) \,\middle|\, f \in \mathcal{F}^T, \operatorname{supp} f(t) \subset \sigma\right\} \subset \mathcal{H}$$

is said to be *reachable* (from $\sigma$, at the moment $t = \xi$). We denote by $P_\sigma^\xi$ the (orthogonal) projection in $\mathcal{H}$ onto $\operatorname{clos}\mathcal{U}_\sigma^\xi$. By definition, $\sigma' \subseteq \sigma''$, $\xi' \leqslant \xi''$ implies $P_{\sigma'}^{\xi'} \leqslant P_{\sigma''}^{\xi''}$.

The family of projections

$$\mathcal{P} := \left\{P_\sigma^\xi \,\middle|\, \sigma \in \Sigma_\Gamma^+,\ \xi \in [0,T]\right\}$$

plays a central role in the further considerations. It determines a (real) Neumann algebra

$$\mathcal{N} := \text{s-clos}\bigvee \mathcal{P} \subset \mathfrak{B}(\mathcal{H})$$

consisting of self-adjoint operators.

An operator in $\mathcal{H}$ of the form $\widehat{\tau}_\sigma := \int_0^T \xi\, dP_\sigma^\xi$ is called an *eikonal*.[10] The eikonals determine the subalgebra

$$\mathcal{T} := \left[\text{u-clos}\bigvee \left\{\widehat{\tau}_\sigma \,\middle|\ \sigma \in \Sigma_\Gamma^+,\ \xi \in [0,T]\right\}\right] \cup \mathbb{I} \ \subset\ \mathfrak{B}(\mathcal{H})$$

which is also determined by the family $\mathcal{P}$. By the well-known properties of the operator integral $\displaystyle\int_0^T g(\xi)\, dP_\sigma^\xi$, the algebra $\mathcal{T}$ is $s$-dense in $\mathcal{N}$, which enables one to use it as a supporting algebra of $\mathcal{N}$.

Along with the family $\mathcal{P}$, the pair $\{\mathcal{N}, \mathcal{T}\}$ is a well-defined intrinsic object of any DSBC $\alpha^T$ belonging to the above-introduced class. It can be trivial: for instance, in the analogue of system $\alpha^T_{\text{Riem}}$ governed by the heat conductivity equation, owing to the infiniteness of the domain of dependence, one has $P_\sigma^\xi = \mathbb{I}$ for any $\xi > 0$ (G. Lebeau, L. Robbiano; 1994). Then, in this case, $\mathcal{N} = \mathcal{T} = \{c\,\mathbb{I} \,|\, c \in \mathbb{R}\}$. However, the pair $\{\mathcal{N}, \mathcal{T}\}$ does always exist.

**4.4. Hyperbolicity.** We will deal with the following class of systems.

A system $\alpha^T$ is said to be *hyperbolic* (write $\alpha^T \in \text{Hyp}$) if

[9] $T - \xi$ is the value of delay, $\xi$ is an action time.

[10] This term will be motivated later.

1. *Continuity.* Every $P_\sigma^\xi \in \mathcal{P}$ is continuous with respect to $\xi$: $\lim\limits_{\xi\to\xi'} P_\sigma^\xi = P_\sigma^{\xi'}$ (in the sense of $s$-convergence).
2. *Commutativity.* $P_{\sigma'}^{\xi'} P_{\sigma''}^{\xi''} = P_{\sigma''}^{\xi''} P_{\sigma'}^{\xi'}$ for all projections in $\mathcal{P}$.
3. *Cyclicity.* $\mathcal{P}$ possesses a cyclic element in $\mathcal{H}$.
4. *Exhausting property.* $P_\sigma^T = \mathbb{I}$ for all $\sigma$.

Property 1 is principal: the continuously extending reachable sets correspond to the intuitive image of the waves propagating with finite speed. By properties 2 and 3, the algebra $\mathcal{N}$ is a commutative cyclic Neumann algebra, which Theorem 2.3 can be applied to. Property 4 is rather technical and is accepted just for the sake of simplicity: it describes the case where the waves propagate into a bounded domain and exhaust the inner space for sufficiently large times.

In addition, we note that, by continuity, each eikonal

$$\widehat{\tau}_\sigma = \int_0^T \xi \, dP_\sigma^\xi$$

has a purely continuous spectrum filling the segment $[\,0, \|\widehat{\tau}_\sigma\|\,] \subseteq [0, T]$.

**4.5. Geometrization.** The reason to single out the class Hyp is that any hyperbolic system can be represented (realized) in a canonical form possessing all features of "intuitive hyperbolicity."

Let $\alpha^T$ be a hyperbolic system, and let $\mathcal{N}$ and $\mathcal{T}$ be the operator algebras of $\alpha^T$ defined in Subsect. 4.3.

*Step* 1. Find the spectrum $\Omega_{\mathcal{T}} =: \Omega$[11] and apply the G-transform to $\mathcal{T}$. The transform maps $\mathcal{T}$ onto $\mathcal{C}(\Omega)$. The eikonals $\widehat{\tau}_\sigma \in \mathcal{T}$ are transferred to the functions $\tau_\sigma(\cdot) := G\widehat{\tau}_\sigma$, called *eikonals.*

By the definition of a supporting algebra, $\{\widehat{\tau}_\sigma\} \cup \mathbb{I}$ generates $\mathcal{T}$. Hence the set of eikonals $\{\tau_\sigma(\cdot)\}$ supplemented with the unit function $1 = G\mathbb{I}$ generates $\mathcal{C}(\Omega)$.

Each eikonal $\tau_\sigma$ is a continuous function on $\Omega$. By the well-known property of G-transform (see, for example, [9]), the values of $\tau_\sigma$ cover the segment $\operatorname{spec}\widehat{\tau}_\sigma = [\,0, \|\widehat{\tau}_\sigma\|\,] \subseteq [0, T]$. Eikonals determine the *subdomains* $\Omega_\sigma^\xi := \{x \in \Omega \mid \tau_\sigma(x) \leqslant \xi\}\,,\ \xi \geqslant 0$.

Denote $\widetilde{\sigma} := \Omega_\sigma^0 = \{x \in \Omega |\ \tau_\sigma(x) = 0\}$ and $\widetilde{\Gamma} := \Omega_\Gamma^0$. These definitions determine a map $\Gamma \supseteq \sigma \mapsto \widetilde{\sigma} \subseteq \widetilde{\Gamma}$ from $\Sigma_\Gamma^+$ into a family of the closed subsets on $\widetilde{\Gamma}$.[12]

[11] This is a key point: the future "wave-guide set" $\Omega$ is appearing!

[12] Under some additional assumptions in the definition of Hyp, one could make this map to be a bijection between $\Gamma$ and $\widetilde{\Gamma}$.

*Step* 2. Choose a cyclic element $h \in \mathcal{H}$ and construct a unitary transform $U$ that diagonalizes the algebra $\mathcal{N}$ (see Subsect. 4.1) and represents the inner space $\mathcal{H}$ in the form $L_{2,\,\mu}(\Omega) = \widetilde{\mathcal{H}} := U\mathcal{H}$.

Thus, along with the original system $\alpha^T$, we get its isometric copy (realization) $\widetilde{\alpha}^T$ determined by $\{\Gamma, \lambda; \widetilde{\mathcal{H}}; \widetilde{W^T}\}$ and governed by the control operator $\widetilde{W^T} : \mathcal{F}^T \to \widetilde{\mathcal{H}}$, $\widetilde{W^T} := UW^T$ (see the right upper corner of the diagram in Fig. 2).

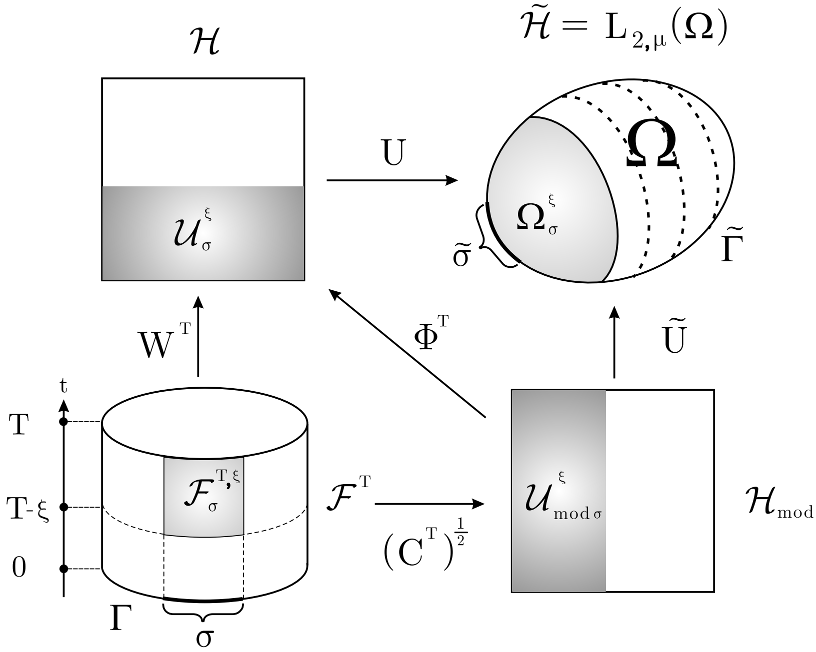


**Fig. 2** Geometrization of $\alpha^T$.

By isometry, the copy is a hyperbolic system. The following properties of $\widetilde{\alpha}^T$ motivate our notion of "hyperbolicity."

By the definition of $U$ (see Subsect. 3.1),

$$U\left[\int_0^T \xi\, dP_\sigma^\xi\right]U^* = U\widehat{\tau}_\sigma U^* = \mathrm{ad}\ \tau_\sigma = \int_0^T \xi\, d[\mathrm{ad}\ \chi_\sigma^\xi]\,,$$

where $\chi_\sigma^\xi$ is the indicator (characteristic function) of $\Omega_\sigma^\xi$ and the last equality is the well-known representation of a continuous multiplicator in the form of an operator integral. By standard functional calculus rules, for a function $g = g(s)$, $s \in \mathbb{R}$, we have

$$U\left[\int_0^T g(\xi)\, dP_\sigma^\xi\right] U^* = \text{ ad } [g \circ \tau_\sigma] = \int_0^T g(\xi)\, d[\text{ad } \chi_\sigma^\xi]\,.$$

In particular, taking $g$ equal to the indicator of $[\xi, T]$, we arrive at $\widetilde{P}_\sigma^\xi := UP_\sigma^\xi U^* = \text{ad } \chi_\sigma^\xi$. Thus, the projection $\widetilde{P}_\sigma^\xi$ cuts off functions on the subdomain $\Omega_\sigma^\xi$.

Correspondingly, for the reachable sets $\widetilde{\mathcal{U}}_\sigma^\xi := \widetilde{W}^T \mathcal{F}_\sigma^{T,\,\xi} = U\mathcal{U}_\sigma^\xi$ of the system $\widetilde{\alpha}^T$ we have

$$\text{clos}\widetilde{\mathcal{U}}_\sigma^\xi = U\text{clos}\mathcal{U}_\sigma^\xi = UP_\sigma^\xi \mathcal{H} = \widetilde{P}_\sigma^\xi\, U\mathcal{H} = [\text{ad } \chi_\sigma^\xi]\, L_{2,\,\mu}(\Omega)\,.$$

Hence $\widetilde{\mathcal{U}}_\sigma^\xi$ is supported in $\Omega_\sigma^\xi$ and is dense in $L_{2,\,\mu}(\Omega_\sigma^\xi)$.

Let us summarize the aforesaid. Take a control $f \in \mathcal{F}^T$ such that $\operatorname{supp} f(t) \subset \sigma$. Let $\widetilde{u}^f(\cdot, t) := U[u^f(t)]$, $t \in [0, T]$, be the corresponding trajectory of $\widetilde{\alpha}^T$. For fixed $t = \xi$ we have $\widetilde{u}^f(\cdot, \xi) \in \widetilde{\mathcal{U}}_\sigma^\xi$. As was shown, this leads to $\operatorname{supp} \widetilde{u}^f(\cdot, \xi) \subset \Omega_\sigma^\xi$. Thus, in the process of evolution, a “wave” $\widetilde{u}^f(\cdot, t)$ appears at the set $\widetilde{\sigma} = \Omega_\sigma^0$ and, as $t$ grows, fills the increasing subdomains $\Omega_\sigma^t$, i.e., propagates from $\widetilde{\sigma}$ into $\Omega$ with finite speed. Loosely speaking, the passage from the original system to its $U$-image visualizes the evolution. Therefore, $\widetilde{\alpha}^T$ is referred to as a *geometrization* of $\alpha^T$.

**4.6. More about the example.** We return to the system $\alpha^T_{\text{Riem}}$ on a manifold. For $\sigma \subset \Gamma := \partial\Omega$ we define a *metric eikonal*[13] (distant function) $d_\sigma :=$ dist$(\,\cdot\,, \sigma) \in \mathcal{C}(\Omega)$ and the family of the closed metric neighborhoods

$$\Omega_\sigma^\xi := \left\{x \in \Omega \,\middle|\, d_\sigma(x) \leqslant \xi\right\}\,, \qquad \xi \geqslant 0\,.$$

Let $\chi_\sigma^\xi(\cdot)$ be the indicator of $\Omega_\sigma^\xi$. Note that these neighborhoods exhaust the manifold: $\Omega_\sigma^T = \Omega$ since $T > \text{diam } \Omega$.

Below we use the representation

$$d_\sigma = \int_0^T \xi\, d[\text{ad } \chi_\sigma^\xi]1$$

which follows from definitions, and its operator version

[13] This term is taken from applications (mechanics, geometrical optics, etc).

$$\text{ad } d_\sigma = \int_0^T \xi \, d[\text{ad } \chi_\sigma^\xi]$$

for the operator multiplying functions by $d_\sigma$.

By the hyperbolicity of the wave equation (4.1), the inclusion $f \in \mathcal{F}_\sigma^{T,\xi}$ leads to $\operatorname{supp} u^f(\cdot, T) \subseteq \Omega_\sigma^\xi$ and hence to $\mathcal{U}_\sigma^\xi \subset L_2(\Omega_\sigma^\xi)$. It is important that the last embedding is dense: by the fundamental Holmgren–John–Tataru uniqueness theorem, the equality $\operatorname{clos} \mathcal{U}_\sigma^\xi = L_2(\Omega_\sigma^\xi)$ holds[14] (see [1]). This equality leads to the coincidence of projections:

$$P_\sigma^\xi = \text{ad } \chi_\sigma^\xi \,, \tag{4.4}$$

whereas $\Omega_\sigma^T = \Omega$ implies $P_\sigma^T = \mathbb{I}$. It is evident that the family $\mathcal{P} = \{P_\sigma^\xi\}$ possesses properties 1–4 (see Subsect. 4.4) and we see that $\alpha_{\text{Riem}}^T \in \text{Hyp}$.

The operator eikonals of the system $\alpha_{\text{Riem}}^T$ are of the form

$$\widehat{\tau}_\sigma = \int_0^T \xi \, dP_\sigma^\xi = \langle \text{ see } (4.4) \, \rangle = \int_0^T \xi \, d[\text{ad } \chi_\sigma^\xi] = \text{ad } d_\sigma, \tag{4.5}$$

i.e., $\widehat{\tau}_\sigma$ multiplies by a metric eikonal and $\widehat{\tau}_\sigma 1 = d_\sigma$.

A simple geometric fact is that metric eikonals distinguish points of $\Omega$: if $d_\sigma(x') = d_\sigma(x'')$ for all $\sigma$, then $x' = x''$. Therefore, by the Stone–Weierstrass theorem, the family $\{\widehat{\tau}_\sigma 1 | \; \sigma \in \Sigma_\Gamma^+\}$ supplemented with the unit function generates $\mathcal{C}(\Omega)$. Turning to the pair $\{\mathcal{N}, \mathcal{T}\}$ of the system $\alpha_{\text{Riem}}^T$, we find $\mathcal{T} = \text{ad } \mathcal{C}(\Omega)$.

The last relation implies $\mathcal{N} = \text{s-clos}\, \mathcal{T} = \text{s-clos}\, \text{ad } \mathcal{C}(\Omega) = \text{ad } L_\infty(\Omega)$. Hence the pair $\{\mathcal{N}, \mathcal{T}\}$ of the system $\alpha_{\text{Riem}}^T$ is $\{\text{ad } L_\infty(\Omega), \text{ad } \mathcal{C}(\Omega)\}$.

Realizing $\alpha_{\text{Riem}}^T$ in the canonical form in accordance with the scheme of Subsect. 4.5 (i.e., diagonalizing $\mathcal{N}$), we first find the G-transform of $\mathcal{T} = \text{ad } \mathcal{C}(\Omega)$. By Theorem 2.2, the image $G\mathcal{T}$ is identical to $\mathcal{T}$ (namely, identifying $\Omega \ni x_0 \equiv \delta_{x_0} \in \Omega_{\mathcal{T}}$, we get $Ga \equiv a$). Further, choosing $h = 1 \in \mathcal{H}$ as a cyclic element of $\mathcal{N}$, we get $U \equiv \mathbb{I}$, which leads to $\tau_\sigma = U[\widehat{\tau}_\sigma 1] \equiv \widehat{\tau}_\sigma 1 = \langle \text{see } (4.5) \rangle = d_\sigma$ and justifies the term "eikonal" applied to the function $\tau_\sigma$ in advance. As a consequence of the identity $G\mathcal{T} \equiv \mathcal{T}$, we get the identity of trajectories: $\widetilde{u}^f = U u^f \equiv u^f$.

Summarizing the aforesaid, we conclude that the system $\alpha_{\text{Riem}}^T$ is identical to its geometrization $\widetilde{\alpha}_{\text{Riem}}^T$.

**4.7. Inverse problem.** We return to the general case of DSBC $\alpha^T$ (see Subsect. 4.2) and associate with it a collection $\{\Gamma, \lambda; C^T\}$ referred to as the

[14] In control theory, this equality is interpreted as the *local approximate boundary controllability* of the system $\alpha_{\text{Riem}}^T$: any function supported in the subdomain $\Omega_\sigma^\xi$ filled with waves propagating from $\sigma$, can be approximated (in $L_2$-metric) with a wave generated at $\sigma$.

*inverse data*. A remarkable fact is that these data determine the system up to an isometry of its inner space.

Indeed, representing the control operator in the form of the polar decomposition, we have $W^T = \Phi^T|W^T|$, where $|W^T| := \left[(W^T)^*W^T\right]^{\frac{1}{2}} = (C^T)^{\frac{1}{2}}$ and $\Phi^T$ is an isometry from $\mathcal{F}^T$ onto $\mathcal{H}_{\rm mod} := \operatorname{clos}\operatorname{Ran}|W^T| \subset \mathcal{F}^T$. The collection $\{\Gamma, \lambda; \mathcal{H}_{\rm mod}; |W^T|\}$ determines a DSBC $\alpha^T_{\rm mod}$ whose trajectories are connected with the trajectories of the original $\alpha^T$ through an isometry: $u^f_{\rm mod}(t) = (\Phi^T)^*[u^f(t)]$, $t \in [0,T]$. We say that the system $\alpha^T_{\rm mod}$ is a *model* of the system $\alpha^T$. The operator $|W^T| = (C^T)^{\frac{1}{2}}$ plays the role of its control operator and its reachable sets are $\mathcal{U}^\xi_{{\rm mod}\,\sigma} = (C^T)^{\frac{1}{2}}\mathcal{F}^{T,\xi}_\sigma \subset \mathcal{H}_{\rm mod}$ (we denote by $P^\xi_{{\rm mod}\,\sigma}$ the corresponding projections). The connecting operators of the original and the model automatically coincide (see the right lower corner in Fig. 2).

At the given level of generality, the model solves the inverse problem consisting in recovering a DSBC from its inverse data. More precisely, constructing the model, we obtain a system possessing the prescribed inverse data.

The inverse data of a hyperbolic $\alpha^T$ determine its geometrization $\widetilde{\alpha}^T$. Indeed, $\alpha^T \in \operatorname{Hyp}$ implies $\alpha^T_{\rm mod} \in \operatorname{Hyp}$, whereas $\widetilde{\alpha}^T$ and $\widetilde{\alpha}^T_{\rm mod}$ are identical just by the isometry "system $\leftrightarrow$ model" realized by the operator $\Phi^T$. In other words, geometrizing the model (see the transform $\widetilde{U}$ in Fig. 2), we obtain the geometrization of the original.

For any hyperbolic $\alpha^T$ (see Subsect. 4.4)

- $C^T$ is a positive operator in $\mathcal{F}^T$,
- the family of projections $\mathcal{P}_{\rm mod} = \{P^\xi_{{\rm mod}\,\sigma} \mid \sigma \in \Sigma^+_\Gamma,\ \xi \in [0,T]\}$ possesses properties 1–4.

It is easy to see that the same conditions are sufficient for $\{\Gamma, \lambda; C^T\}$ to be the inverse data of a hyperbolic system (namely, of $\alpha^T_{\rm mod}$). So, we obtain conditions characterizing the data; moreover, to improve or simplify this characterization at such a level of generality is hardly possible.

**4.8. Determination of manifolds.** Recall two traditional settings of the inverse problems for the system $\alpha^T_{\rm Riem}$.

The "input → output" correspondence of the system (4.1)–(4.3) is realized by a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $R^T f := u^f|_{\Gamma\times[0,T]}$. The *dynamical* IP is to recover the manifold $\Omega$[15] from its response operator. Because of the finiteness of the wave propagation speed, the response operator has to be given for large enough $T$ (see [1]) and, just for the sake of simplicity, we assume that $R^{2T}$ is known for fixed $T > \operatorname{diam}\Omega$.

The second problem is formulated as follows. Let $\{\lambda_k\}_{k=0}^\infty : 0 = \lambda_0 \leqslant \lambda_1 \leqslant \ldots$ and $\{\varphi_k\}_{k=0}^\infty : (\varphi_i, \varphi_j)_{\mathcal{H}} = \delta_{ij}$ be the spectrum and the normalized eigenfunctions of the Neumann spectral problem

[15] i.e., to determine $\Omega$ up to isometry of Riemannian manifolds

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi && \text{in } \Omega \setminus \Gamma, \\ \nu \cdot \nabla\varphi &= 0 && \text{on } \Gamma\,. \end{aligned}$$

The set $\{\lambda_k; \varphi_k|_\Gamma\}_{k=0}^\infty$ is called the (*Neumann*) *spectral data* of the manifold $\Omega$. The *spectral* IP is to recover $\Omega$ from its spectral data.

Both IPs are solved by the boundary control method (see [1, 3]). From the viewpoint of this paper, the role of the traditional data is just to compose the connecting operator. Namely, we have the representations

$$C^T \;=-\frac{1}{2}\,(S^T)^* R^{2T} J^{2T} S^T \;=\; \sum_{k=0}^{\infty} (\,\cdot\,, s_k^T)_{\mathcal{F}^T}\, s_k^T\,,$$

where the map $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0,T]$ to $\Gamma \times [0,2T]$ as odd functions of $t$ with respect to $t = T$; $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is the integration $(J^{2T} f)(\cdot,t) := \int\limits_0^t f(\cdot,s)\,ds$. In the second representation,

$$s_k^T = s_k^T(\gamma,t) := (\lambda_k)^{-\frac{1}{2}} \sin\left[(\lambda_k)^{\frac{1}{2}}(T-t)\right] \varphi_k(\gamma)$$

are the functions on $\Gamma \times [0,T]$[16] (see [1, 3]).

Thus, both dynamical and spectral data determine the collection $\{\Gamma, \mathrm{vol}_\Gamma; C^T\}$, which suffices for constructing the model of $\alpha^T_{\mathrm{Riem}}$ and its further geometrization. Since, in this case, the geometrization is identical to the original, the manifold $\Omega$ is recovered (as a cHs, up to a homeomorphism). Extra efforts are required to endow it with the Riemannian structure, but these details are of technical character and we refer the reader to papers on the BC-method.

**4.9. Comments.** 1. The assumption $T > \mathrm{diam}\,\Omega$ is accepted for simplicity: in fact, given $R^{2T}$ for a fixed $T > 0$, the BC-method recovers the submanifold $\Omega^T_\Gamma$ (see [1]). However, such a *time optimal* determination can be also implemented in terms of geometrization.

2. For systems on manifolds, the following "characterization" of the dynamical data can be proposed. An operator $R^{2T}$ acting in the space $\mathcal{F}^{2T}$ is the response operator of a system $\alpha^T_{\mathrm{Riem}}$ if and only if the operator $-\frac{1}{2}\,(S^T)^* R^{2T} J^{2T} S^T$ satisfies the conditions formulated at the end of Subsect. 4.7. Surely, such a description can be hardly regarded to as efficient and checkable, but we are rather sceptical of the possibility to obtain something better. One more version of the characteristic conditions proposed in [2] also uses $P^\xi_{\mathrm{mod}\,\sigma}$. Then the following question arises: Whether it is possible to characterize the data in terms of $C^T$ in itself, not invoking these projections. This question is still open.

[16] $s_0^T$ is understood as $(T-t)\,\varphi_0|_\Gamma$

3. The commutativity and cyclicity of the family $\mathcal{P}$ is a specific feature of systems governed by *scalar* equations like (4.1), whose states are scalar functions. In more complicated case of systems with vector states (electrodynamics, elasticity theory), the structure of the reachable sets, as well as properties of the corresponding projections $P^{\xi}_{\sigma}$, are poorly known. Anyway, the cyclicity certainly does not hold and it is not reasonable to hope that commutativity holds either. Most probably, the relevant algebra $\mathcal{N}$ is not commutative, its spectrum is not a well-defined object,[17] and geometrization becomes problematic. This rises the following "philosophical" question: In the case of vector systems, where can a cHs $\Omega$ appear from? As in the case of the higher-dimensional Calderon problem (see Subsect. 3.7), the question is open. A unified answer in terms of functional analysis could be evaluated as a substantial progress in this area of IPs.

**Acknowledgement.** I am grateful to V.M. Isakov for his kind invitation to write this paper. I would like to thank S.V. Belisheva and I.V. Kubyshkin for the help in computer graphics. The work is supported by the Russian Foundation for Basic Research (grants No. 08-01-00511 and NSh-8336.2006.1).

## References

1. Belishev, M.I.: Boundary control in reconstruction of manifolds and metrics (the BC-method). Inverse Probl. **13**, no. 5, 1-45 (1997)
2. Belishev, M.I.: Dynamical systems with boundary control: models and characterization of inverse data. Inverse Probl. **17**, 659-682 (2001)
3. Belishev, M.I.: Recent progress in the boundary control method. Inverse Probl. **23**, no. 5, 1-67 (2007)
4. Belishev, M.I.: The Calderon problem for two-dimensional manifolds by the BC-method. SIAM J. Math. Anal. **35**, no. 1, 172-182 (2003)
5. Belishev, M.I., Sharafutdinov, V.A.: Dirichlet to Neumann operator on differential forms. Bull. Sci. Math. **132**, no. 2, 128-145 (2008)
6. Forster, O.: Lectures on Riemann surfaces. Springer-Verlag, New-York-Heidelberg-Berlin (1981)
7. Henkin, G., Michel, V.: On the explicit reconstruction of a Riemann surface from its Dirichlet–Neumann operator. Geom. Funct. Anal. **17** no 1, 116-155 (2007)
8. Lassas, M., Uhlmann, G.: On determining a Riemann manifold from the Dirichlet-to-Neumann map. Ann. Sci. Ec. Norm. Sup. **34**, no. 5, 771-787 (2001)
9. Murphy, G.J.: $C^*$-Algebras and Operator Theory. Academic Press, San Diego (1990)
10. Naimark, M.A.: Normed Rings. WN Publishing, Gronnongen, The Netherlands (1970)
11. Schwarz, G.: Hodge Decomposition - a Method for Solving Boundary Value Problems. Springer-Verlag, Berlin , (1995)
12. Shafarevich, I.R.: Basic Notions of Algebra (in Russian). RC, Moscow, Izhevsk (2001)
13. Sobolev, S.L.: Some Applications of Functional Analysis in Mathematical Physics (in Russian). 1st ed. Leningrad State Univ., Leningrad (1950); 3rd ed. Nauka, Moscow

[17] However, certain candidates for this role do exist: see [9].

(1988); English transl. of the 1st Ed.: Am. Math. Soc., Providence, RI (1963); English transl. of the 3rd Ed. with comments by V. P. Palamodov: Am. Math. Soc., Providence, RI (1991)

# The Ginzburg-Landau Equations for Superconductivity with Random Fluctuations

Andrei Fursikov, Max Gunzburger, and Janet Peterson

*Dedicated to the memory of Sergey L'vovich Sobolev,*
*one of the greatest mathematicians of the twentieth century*

**Abstract** Thermal fluctuations and material inhomogeneities have a large effect on superconducting phenomena, possibly inducing transitions to the non-superconducting state. To gain a better understanding of these effects, the Ginzburg–Landau model is studied in situations for which the described physical processes are subject to uncertainty. An adequate description of such processes is possible with the help of stochastic partial differential equations. The boundary value problem of Neumann type for the stochastic Ginzburg–Landau equations with additive and multiplicative white noise is investigated. We use white noise with minimal restriction on its independence property. The existence and uniqueness of weak and strong statistical solutions are proved. Our approach is based on using difference schemes for the Ginzburg–Landau equations.

## 1 Introduction

This paper is dedicated to the memory of Sergey L'vovich Sobolev. His outstanding contributions to the theory for the equations of mathematical physics are extremely deep and influential. Indeed, since the 1960s, practically all investigations in the aforementioned field of mathematics use Sobolev

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series.*
doi: 10.1007/978-0-387-85652-0, 

spaces and, at the least, are thereby guided by Sobolev's ideas. The present paper, of course, is no exception to this common rule. Moreover, the use of Sobolev spaces in complicated functional constructions for stochastic partial differential equations is especially successful and effective. Note also that being the closest aide to I.V. Kurchatov in the realization of the nuclear project in the Soviet Union after 1943, S.L. Sobolev took part in the numerical solution of huge problems of mathematical physics. From that time on to the end of his life, he had an invariable interest in the discrete approximation of continuum objects, especially in cubature formulas. In the present paper, discrete approximations are not only used, they play a crucial role in obtaining the main results.

This paper is devoted to the mathematical study of a boundary value problem for the stochastic Ginzburg–Landau model of superconductivity; we hope it will promote a better understanding of the transitions that occur between the superconducting and nonsuperconducting states.

In 1908, Kamerlingh–Onnes discovered that when metals such as mercury, lead, and tin are cooled to an absolute temperature below some small but positive critical value, their electrical resistivity completely disappears. This was a great surprise since what was expected is that the resistivity of metals would smoothly tend to zero as the temperature also tended to zero. In addition to this *zero resistance* property, superconductors are characterized by the property of *perfect diamagnetism*. This phenomenon was discovered in 1933 by Meissner and Ochsenfeld and is also known as the Meissner effect. What they observed is that not only is a magnetic field excluded from a superconductor, i.e., if a magnetic field is applied to a superconducting material at a temperature below the critical temperature, it does not penetrate into the material, but also that a magnetic field is expelled from a superconductor, i.e., if a superconductor subject to a magnetic field is cooled through the critical temperature, the magnetic field is expelled from the material. One of the consequences of the Meissner effect is that superconductors cannot be "perfect conductors" which are idealized (and unattainable) materials that have zero resistivity and that can be described by the linear Maxwell equations of electromagnetism. For such materials the magnetic field would not be expelled from the material when it is cooled through the critical temperature.

Superconductivity was not adequately explained until, in 1957, Bardeen, Cooper, and Schrieffer (BCS) [1] published their landmark paper describing a microscopic theory of superconductivity. However, even earlier, several phenomenological *continuum* theories were proposed, most notably by Ginzburg and Landau [20] in 1950. The Ginzburg–Landau theory was itself based on a general theory, introduced by Landau in 1937, for second-order phase transitions in fluids. Ginzburg and Landau thought of conducting electrons as being a "fluid" that could appear in two phases, namely superconducting and normal (non-superconducting). Through a stroke of intuitive genius, Ginzburg and Landau added to the theory of phase transitions certain effects, motivated by quantum-mechanical considerations, to account for how the electron "fluid" motion is affected by the presence of magnetic fields. In 1959, Gor'kov [21] showed that, in an appropriate limit, the macroscopic Ginzburg–Landau theory can be derived from the microscopic BCS theory. Details about the Ginzburg–Landau model can be found in [7, 13, 12, 41], the last of which may also be consulted for details about the BCS model.

The dependent variables of the Ginzburg–Landau model are the complex-valued order parameter $\psi$ and the vector-valued magnetic potential $A$. Physically interesting variables such as the density of superconducting electrons, the current, and the induced magnetic field can be easily deduced from $\psi$ and $A$. The Ginzburg–Landau model itself can be expressed as a system of two coupled partial differential equations from which $\psi$ and $A$ can be determined. One of these equations is a vector-valued, nonlinear Maxwell equation that relates the supercurrent, i.e., the current that flows without resistance, to a nonlinear function of $\psi$, $\nabla\psi$, and $A$. The second equation is a complex-valued equation that relates spatial and temporal variations of $\psi$ to a nonlinear potential energy term. After appropriate non-dimensionalizations, there are two non-dimensional parameters appearing in the differential equations. One is the ratio of the relaxation times of $\psi$ and $A$, the other, known as the Ginzburg–Landau parameter, is the ratio of the characteristic lengths over which $\psi$ and $A$ vary. These two length scales are referred to as the coherence and penetration lengths respectively.

In this paper, we consider a simplified Ginzburg–Landau system for $\psi$ in which $A$ is assumed to be a given vector-valued field. There are two situations of paramount practical interest for which the use of this simplified Ginzburg–Landau system can be justified. First, for high values of the Ginzburg–Landau parameter, it can be shown [6, 12] that, to leading order, the magnetic field in a superconductor is simply that given by the linear Maxwell equations so that $A$ may be determined from these equations. Thus, insofar as the other component equation of the Ginzburg–Landau model is concerned, $A$ can be viewed as a given vector field. A similar uncoupling can be shown to occur for thin film samples [5] for all values of the Ginzburg–Landau parameter. Most superconductors of practical interest are characterized by "high" values of the Ginzburg–Landau parameter and superconducting films are of very substantial technological interest; the simplified Ginzburg–Landau system we study can be used to model both of these situations. Furthermore, in the more general case where one has to consider the fully coupled Ginzburg–Landau equations for $\psi$ and $A$, random fluctuations enter into the system in very much the same way as they do for the simplified system, so much of what is learned about stochastic versions of the simplified system applies to stochastic versions of the full system.

The Ginzburg–Landau theory is applicable only to highly idealized physical contexts that do not take into account factors such as material inhomogeneities and thermal fluctuations due to molecular vibrations. Both these factors play a crucial role in practical superconductivity since the former enables large currents to flow through a superconductor without resistance while the latter can have the opposite effect, especially at temperatures close to critical transition temperature (see, for example, [30, 39]). In [22], it is shown that, within the Ginzburg–Landau framework, thermal fluctuations are properly modeled by an additive white noise term in the Ginzburg–Landau equation for $\psi$; the amplitude of the noise term grows as the temperature approaches the critical temperature. In [4, 30], it is shown that, again in the Ginzburg–Landau framework, material inhomogeneities can be correctly modeled through the coefficient of the linear (in $\psi$) term in the Ginzburg–Landau equation for $\psi$; random variations in the material properties can thus be modeled as random perturbations in this coefficient which results in a multiplicative white noise term in the Ginzburg–Landau equations. In this paper, we treat both the additive and multiplicative noise cases. Studies of the physics of superconductors in the presence of white noise perturbations can be found in [11, 15, 23, 35, 39, 42, 43]; computational studies of the Ginzburg–Landau equations with additive and multiplicative noise are given in [9, 10].

In this paper, we study the stochastic Ginzburg–Landau equation written in the following dimensionless form:

$$d\psi(t,x) + \big((i\nabla + A(x))^2\psi - \psi + |\psi|^2\psi\big)dt = \widehat{r}[\psi]dW, \quad t > 0,\ x \in G \subset \mathbb{R}^d\,, \tag{1.1}$$

where $G$ is a bounded domain, $d = 2, 3$, and an explanation of the notation employed on the right-hand side of (1.1) is given below in (1.3) and (1.4). On the boundary $\partial G$ of $G$, we set

$$\big(i\nabla + A(x)\big)\psi(t,x)\cdot n = 0, \quad t > 0,\ x \in \partial G\,, \tag{1.2}$$

where $n$ denotes the unit outer normal vector to $\partial G$.

From the view of the general theory of dynamical systems, the superconducting state is a stable steady-state solution of (1.1) (with zero right-hand side). The disappearance of the superconducting state (when some parameter of the system changes) means that some other steady-state solution of (1.1) arises and becomes stable or either time-periodic or chaotic behavior is realized.

We emphasize that when the dynamical system became unstable, the classical derivation of the equation for the superconducting state, rigorously speaking, looses its correctness. Indeed, in that derivation, as well as in other derivations of such a kind, only the main "forces" controlling the situation are taken into account and all relatively small and unessential "forces" are omitted, implicitly assuming stability in the sense that small fluctuations of "forces" lead to small fluctuations of the state. In unstable situations, this argument is evidently incorrect. The alternative is to replace, in the unstable situation, all small and unessential "forces" by white noise forcing (additive white noise) or perhaps by white noise multiplied by a function proportional to the state (multiplicative white noise). The physical basis of this approach is that, since "values" of white noise at different times are statistically independent, white noise renders a "smoothing" influence on the dynamical system. In more rigorous terms, this means the addition of white noise to the right-hand side of (1.1) leads to the substitution of many steady-state solutions of (1.1) by the unique (ergodic) statistical steady-state solution of (1.1) that is stable, i.e., that satisfies the *mixing property.* We also note that, in stable situations, replacing unessential "forces" by additive (multiplicative) white noise means taking into account thermal (material inhomogenety) fluctuations, as was noted above.

Very important arguments that can be used to justify the physical adequateness of the aforementioned modeling of superconductivity effects with the help of the stochastic problem (1.1) and (1.2) are given by recent results about ergodicity for abstract dynamical systems, including the two-dimensional Navier–Stokes and Ginzburg–Landau equations with random kick forces or additive white noise. The first results in this direction were obtained in [14, 16, 29]. In these papers, ergodicity was proved in stable situations, i.e., when the corresponding dynamical system with random forces omitted is stable. In the case of an unstable dynamical system, ergodicity was established in [36, 37, 38]. A detailed exposition of this topic can be found in [28].

Taking into account all of the above discussion, the following plan for the mathematical investigation of the superconducting state and its possible disappearance in industrial conditions is possible.

- Proof of the existence and uniqueness of weak and strong solutions of the stochastic boundary value problem (1.1) and (1.2).
- Proof of the ergodicity property for the random dynamical system generated by (1.1) and (1.2).
- Investigation of the disappearance of the supercoducting state in terms of the ergodic measure $P$ that corresponds to the stochastic problem (1.1) and (1.2).

This paper is devoted to the proof of the first of these assertions.

The list of investigations of stochastic parabolic partial differential equations is huge because equations of such type arise in many problems of mathematics, physics, biology, and other applications. Here, we cite only the earliest papers in this field and papers closely connected with our paper. Investigations of linear parabolic stochastic partial differential equations were begun in the middle of 1960s [8]. Nonlinear stochastic parabolic equations were studied in [2, 33] and the stochastic Navier–Stokes system was studied in [3, 44, 45]. The paper [27] and the book [34] contain many deep results on these topics as well as a detailed historical review. Lastly, we note the works [25, 32].

In this paper, we study the stochastic boundary value problem (1.1) and (1.2) for the Ginzburg–Landau equation. Note that the right-hand side in

(1.1) should be written in a more detailed way as follows:

$$\widehat{r}[\psi]dW = r\big(\mathrm{Re}\psi(t,x)\big)\, d\mathrm{Re}W(t,x) + ir\big(\mathrm{Im}\psi(t,x)\big)\, d\mathrm{Im}W(t,x), \qquad (1.3)$$

where $dW = dW(t,x)$ is a complex-valued white noise and, as usual, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real and imaginary parts of a complex number $z$ respectively. In addition, $r(\lambda)$, $\lambda \in \mathbb{R}$, is, roughly speaking,[1] the following function:

$$r(\lambda) = \max\,(\rho_1,\, \rho_2|\lambda|), \qquad \rho_1 > 0,\, \rho_2 \geqslant 0. \qquad (1.4)$$

In particular, when $\rho_2 = 0$, (1.3) reduces to complex-valued additive white noise. Note immediately that the main difficulties we are forced to overcome in this paper are connected with the case $\rho_2 > 0$ which results in some kind of multiplicative white noise. The form (1.3) of the random fluctuations for the Ginzburg–Landau equation is reasonable from our point of view when, describing Ginzburg–Landau flow in instable situation, one replaces all small and unessential "forces" by stochastically independent fluctuations, i.e., by white noise. Indeed, since by the definition of complex-valued white noise $dW(t,x)$, its real ($d\mathrm{Re}W(t,x)$) and imaginary ($d\mathrm{Im}W(t,x)$) parts are mutually independent white noises [19, Chapt. III, Sect. 1]), (1.3) gives the maximal independent form of multiplicative white noise.

In this paper, we provide a detailed exposition of the proof of the existence and uniqueness of weak and strong statistical solutions of the stochastic boundary value problem (1.1) and (1.2). The main feature of our exposition is that, to prove the existence of a weak solution, we use, instead of Galerkin approximations, *approximations by method of lines*, i.e., we introduce a finite difference approximation of the Ginzburg–Landau equation with respect to the spatial variables. Although the method of lines is more complicated in realization than Galerkin's method, it has one important advantage: method of lines approximations inherit the structure of the Ginzburg–Landau equation much better than do Galerkin ones and therefore we can obtain many estimates for method of line approximations that cannot be obtained for Galerkin approximations. All these estimates we essentially use in our proof in order to overcome difficulties arising mostly because of the multiplicative structure of white noise. Nevertheless, one important a priori estimate which can be derived (formally) for the Ginzburg–Landau equation we cannot yet derive for its method of lines approximation. That is why for the three-dimensional Ginzburg–Landau equation with multiplicative white noise, we have proved here only the existence of a weak solution. For the two-dimensional Ginzburg-landau equation with multiplicative white noise as well as for the two- and three-dimensional Ginzburg–Landau equation with additive white noise, we can prove the existence and uniqueness of both weak and strong solutions.

[1] In fact, $r(\lambda)$, is the function (1.4) smoothed at points of discontinuity of its derivative. See the exact definition given below in (3.19).

The structure of the paper can be deduced from its content as described above.

# 2 The Ginzburg–Landau Equation and Its Finite Difference Approximation

In this section, we formulate the boundary value problem for the (simplified) Ginzburg–Landau equations without fluctuations and define an approximation by the method of lines that will play an important role in our analysis.

## *2.1 Boundary value problem for the Ginzburg–Landau equation*

Let $G \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded domain with $C^\infty$-boundary $\partial G$, and let $Q_T = (0, T) \times G$ denote a space-time cylinder. In $Q_T$, we consider the Ginzburg–Landau equation for the complex-valued function $\psi(t, x)$, referred to as the order parameter,

$$\frac{\partial \psi}{\partial t} + \big(i\nabla + A\big)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{for } (t, x) \in Q_T \tag{2.1}$$

along with the boundary condition

$$(i\nabla + A)\psi \cdot n = 0 \quad \text{on } (0, T) \times \partial G \tag{2.2}$$

and the initial condition

$$\psi(0, x) = \psi_0(x) \quad \text{in } G\,, \tag{2.3}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ denotes the gradient operator and $A(x) = (A^1, \dots, A^d)$, the magnetic potential, is a given real-valued vector field such that $\operatorname{div} A = \sum\limits_{j=1}^{d} \frac{\partial A^j}{\partial x_j} = 0$. Also, $n = (n_1, \dots, n_d)$ denotes the unit outer normal vector to the boundary $\partial G$ and $\psi_0(x)$ is a given initial condition. We have

$$\begin{aligned}(i\nabla + A)^2 \psi &= \big(i\nabla + A, i\nabla + A\big)\psi \\ &= \sum_{j=1}^{d} \Big(i\frac{\partial}{\partial x_j} + A^j(x)\Big)\Big(i\frac{\partial \psi(x)}{\partial x_j} + A^j(x)\psi(x)\Big)\,. \end{aligned} \tag{2.4}$$

We assume that $A(x) \in (C^2(\overline{G}))^d$ and, for any fixed time, $\psi(t,x) \in L^2(G)$.

We want to introduce function spaces within which it is natural to look for the solution of the problem (2.1)–(2.3). The Sobolev space of complex-valued functions defined in $G$ and square integrable there together with their derivatives up to order $k$ is denoted by $H^k(G)$, $k \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of positive integers. In addition, we define the space

$$H_A^2(G) = \{\phi(x) \in H^2(G) \ : \ (i\nabla + A)\phi \cdot n = 0 \ \text{ on } \partial G\} . \tag{2.5}$$

The space of solutions of (2.1)–(2.3) is defined as follows:

$$\mathcal{Y} = \Big\{\psi(t,x) \in L^2\big(0,T; H_A^2(G)\big) \cap L^6(Q_T) \ : \ \frac{\partial \psi}{\partial t} \in L^2(Q_T)\Big\} . \tag{2.6}$$

We also study generalized solutions of the problem (2.1)–(2.3). To obtain a weak formulation, we multiply (2.1) by the complex conjugate of $\phi$, denoted by $\overline{\phi}$, and integrate over $Q_T$. Using the boundary condition (2.2) and integration by parts, we obtain

$$\int\limits_{Q_T} \left[\frac{\partial \psi}{\partial t}\overline{\phi} + \big(i\nabla + A\big)\psi \cdot \overline{\big(i\nabla + A\big)\phi} - \psi\overline{\phi} + |\psi|^2\psi\overline{\phi}\right] dxdt = 0 . \tag{2.7}$$

Here, we will not make more precise the function space used for generalized solutions, defined by (2.3) and (2.7) with arbitrary $\overline{\phi} \in L_2(0,T; H^1(G))$, of the problem (2.1)–(2.3) because just at this moment it is not necessary.

## *2.2 Approximation by the method of lines*

The approximation of the solution of a partial differential equation by the method of lines means that we approximate the continuous space variables $x = (x_1, \dots, x_d)$ by a discrete grid or mesh so that we approximate the partial differential equation problem by a system of ordinary differential equations. In our case, we use finite difference quotients to approximate spatial derivatives. We assume that the grid is uniform and the scale of the grid, $h > 0$, is a fixed, sufficiently small number. Let an arbitrary point on the grid be denoted by $kh$, where $k \in \mathbb{Z}^d$, $kh = (k_1 h, \dots, k_d h)$, and $\mathbb{Z}$ denotes the set of integers. Since $\psi(x)$ is a function of the continuous variable $x$, we let $\psi_k$, defined on the given grid, denote the approximation to $\psi$ at the point $kh$.

We now define the corresponding discrete "derivatives" or difference quotients; we distinguish the discrete derivatives from the continuous derivatives $\frac{\partial}{\partial x_j}$ by using the notation $\partial_{j,h}$. Let $\delta_{jk}$ denote the Kronecker delta, and let $e_j = (\delta_{j1}, \dots, \delta_{jd})$, $j = 1, \dots, d$. We can approximate the derivative $\frac{\partial \psi}{\partial x_j}$ by the forward difference quotient $\partial_{j,h}^{+}\psi_k = \frac{1}{h}(\psi_{k+e_j} - \psi_k)$ or by the backward difference quotient $\partial_{j,h}^{-}\psi_k = \frac{1}{h}(\psi_k - \psi_{k-e_j})$. The discrete divergence operator ${\operatorname{div}_h}^{\pm}$, the discrete gradient operator $\nabla_h^{\pm}$, and the discrete Laplace operator $\Delta_h = \operatorname{div}_h^{-}\nabla_h^{+}$ are then defined in an obvious manner.

Analogous to (2.4), we define

$$
\begin{aligned}
(i\nabla_h + A_k)^2\psi_k &= (i\nabla_h^{-} + A_k, i\nabla_h^{+} + A_k)\psi_k \\
&= \sum_{j=1}^{d}\left(i\partial_{j,h}^{-} + A_k^j\right)\left(i\partial_{j,h}^{+}\psi_k + A_k^j\psi_k\right),
\end{aligned}
\tag{2.8}
$$

where $A_k = A(kh)$ and $A_k^j$ denotes the $j$th component of the vector $A_k = (A^1(kh), \dots, A^d(kh))$.

We now approximate the domain $G$ and its boundary $\partial G$.

**Definition 2.1.** The approximate boundary $\partial G_h$ is the subset of the grid $kh$, $k \in \mathbb{Z}^d$, that consists of two parts $\partial G_h = \partial G_h^{+} \cup \partial G_h^{-}$, where

(i) $\partial G_h^{-}$ is the set of points $kh \in G$ such that $(k + e_j)h \in \mathbb{R}^d \setminus G$ or $(k - e_j)h \in \mathbb{R}^d \setminus G$ for some $j = 1, \dots, d$

and

(ii) $\partial G_h^{+}$ the set of points $kh \in \mathbb{R}^d \setminus G$ such that $(k+e_j)h \in G$ or $(k-e_j)h \in G$ for some $j = 1, \dots, d$.

**Definition 2.2.** The approximate domain $G_h$ is the subset of points $kh \in G$, $k \in \mathbb{Z}^d$; we also set $G_h^0 = G_h \setminus \partial G_h^{-}$.

We introduce the following subsets of the approximate boundary $\partial G_h$:

$$
\begin{aligned}
\partial G_h^{+}(-j) &= \{kh \in \partial G_h^{+} \,:\, (k + e_j)h \in \partial G_h^{-}\} \\
\partial G_h^{+}(+j) &= \{kh \in \partial G_h^{+} \,:\, (k - e_j)h \in \partial G_h^{-}\}
\end{aligned}
\quad \text{for } j = 1, \dots, d \tag{2.9}
$$

and

$$
\begin{aligned}
\partial G_h^{-}(-j) &= \{kh \in \partial G_h^{-} \,:\, (k + e_j)h \in \partial G_h^{+}\} \\
\partial G_h^{-}(+j) &= \{kh \in \partial G_h^{-} \,:\, (k - e_j)h \in \partial G_h^{+}\}
\end{aligned}
\quad \text{for } j = 1, \dots, d. \tag{2.10}
$$

The sets $\partial G_h^{+}(\pm j)$ and $\partial G_h^{-}(\pm j)$ are illustrated in Fig. 2.1 for a domain in $\mathbb{R}^2$. In addition, we note that the sets $\partial G_h^{-}(\pm j)$, $j = 1, \dots, d$, can possess nontrivial pairwise intersections.

We now turn to the approximation of the boundary value problem (2.1)–(2.3) by the method of lines. We have

$$
\frac{\partial \psi_k}{\partial t} + (i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k = 0 \quad \text{for } kh \in G_h \tag{2.11}
$$

and

$$
\psi_k\big|_{t=0} = \psi_{0,k} \quad \text{for } kh \in G_h, \tag{2.12}
$$

where the notation $(i\nabla_h + A_k)^2$ is defined by (2.8).

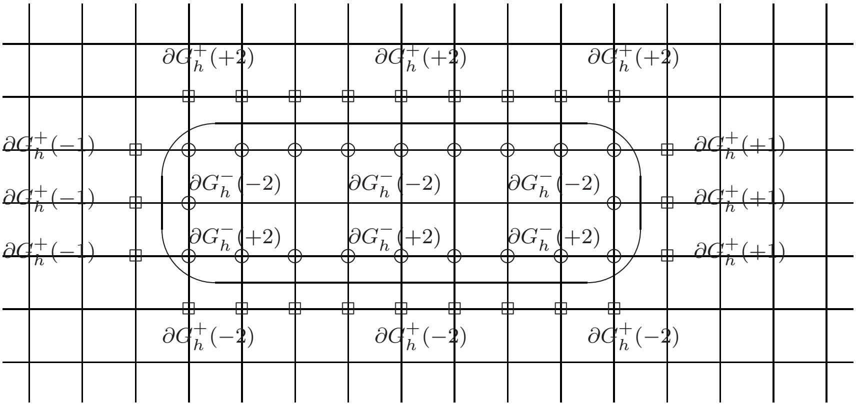


**Fig. 2.1** The approximate boundary $\partial G_h^+$ is denoted by squares, and $\partial G_h^-$ is denoted by circles.

In order to define the analogue of the boundary condition (2.2), we first note that the key property of this condition is that it implies the following formula for integration by parts:

$$\int_G (i\nabla + A)^2\psi(x)\overline{\phi(x)}\,dx = \int_G (i\nabla + A)\psi(x)\overline{(i\nabla + A)\phi(x)}\,dx \qquad \forall\,\psi \in \mathcal{H}_A^2(G), \phi \in \mathcal{H}^1(G)\,. \tag{2.13}$$

Using (2.13), one can define a weak solution of our problem (2.1)–(2.3) with the aid of (2.7). To define the weak solution for the system (2.11) and (2.12), we need the following discrete analogue of (2.13):

$$h^d \sum_{kh\in G_h} \left(i\nabla_h^- + A_k, i\nabla_h^+ + A_k\right)\psi_k\overline{\phi_k}$$
$$= h^d \sum_{j=1}^{d} \sum_{kh\in G_h\cup\partial G_h^+(-j)} (i\partial_{j,h}^+\psi_k + A_k^j\psi_k)\overline{(i\partial_{j,h}^+\phi_k + A_k^j\phi_k)}\,. \tag{2.14}$$

We take this formula, which will be proved in the next subsection, as the foundation for the definition of the discrete analogue of the boundary condition (2.2).

### 2.2.1 Summation by parts formula

In this section, our goal is to prove the discrete analogue of (2.13) given by (2.14).

**Lemma 2.3.** *Let the discrete functions $\phi_k$ and $\psi_k$ be defined for $kh \in G_h \cup \partial G_h^+$. Assume that for each function $\phi_k$*

$$\sum_{j=1}^{d}\Big( \sum_{kh\in\partial G_h^+(-j)} (iV_k^j + hV_k^j A_k^j)\overline{\phi}_k - \sum_{kh\in\partial G_h^+(+j)} iV_{k-e_j}^j \overline{\phi}_k\Big) = 0\,, \tag{2.15}$$

*where*

$$V_k^j = i\frac{\psi_{k+e_j} - \psi_k}{h} + A_k^j \psi_k\,. \tag{2.16}$$

*Then* (2.14) *holds.*

*Proof.* Using (2.16) and setting $r = k - e_j$, we obtain

$$h^d \sum_{kh\in G_h} (i\nabla_h^- + A_k, i\nabla_h^+ + A_k)\psi_k\overline{\phi}_k$$

$$= h^d \sum_{kh\in G_h} \left( \sum_{j=1}^{d} i\frac{V_k^j - V_{k-e_j}^j}{h} + A_k^j V_k^j \right)\overline{\phi}_k$$

$$= h^d \sum_{j=1}^{d}\Bigg[ \sum_{kh\in\partial G_h^-(+j)} \frac{(-i)}{h} V_{k-e_j}^j \overline{\phi}_k + \sum_{kh\in\partial G_h^-(-j)} \left(\frac{i}{h}V_k^j + A_k^j V_k^j\right)\overline{\phi}_k$$

$$+ \sum_{rh\in G_h\setminus\partial G_h^-(-j)} \frac{-i}{h} V_r^j \overline{\phi}_{r+e_j} + \sum_{kh\in G_h\setminus\partial G_h^-(-j)} \left(\frac{i}{h}V_k^j + A_k^j V_k^j\right)\overline{\phi}_k\Bigg]$$

$$= h^{d-1}\sum_{j=1}^{d}\Bigg[ \sum_{kh\in\partial G_h^-(-j)} \left(iV_k^j + hA_k^j V_k^j\right)\overline{\phi}_k - \sum_{rh\in\partial G_h^+(-j)} iV_r^j\overline{\phi}_{r+e_j}\Bigg]$$

$$+ h^d \sum_{j=1}^{d} \sum_{kh\in G_h\setminus\partial G_h^-(-j)} V_k^j \left(\overline{i\frac{\phi_{k+e_j} - \phi_k}{h} + A_k^j\phi_k}\right).$$

We add to the right-hand side of this relation the left-hand side of (2.15), where in the second sum we use the change of variables $r = k - e_j$. After performing this substitution, we arrive at (2.14). □

Thus, the relation (2.15) contains the boundary conditions we need. We only need to write these conditions in a more convenient form.

### 2.2.2 Boundary conditions for the system (2.11)

Since $G \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with $C^\infty$-boundary $\partial G$, at each point $x \in \partial G$ the main curvatures of the surface $\partial G$ are well defined. When $d = 2$, the curve $\partial G$ has at $x \in \partial G$ one main curvature (usually called the curvature). We denote the modulus of this curvature as $\kappa(x)$. When $d = 3$, we denote by $\kappa(x) = \max\{|\kappa_1(x)|, |\kappa_2(x)|\}$, where $\kappa_j(x)$, $j = 1, 2$, are the main curvatures of $\partial G$ at the point $x$. We set

$$\widehat{\kappa} = \max_{x \in \partial G} \kappa(x)\,.$$

We take a ball of radius $r < 1/\widehat{\kappa}$ and touch this ball at any $x \in \partial G$ from each of the two sides of the surface $\partial G$. Decreasing the radius $r$, we can position this ball so that it will not intersect $\partial G$ at any point other than the point $x$ of contact of the ball and $\partial G$. We denote such a radius $r(x)$ by $r_0(x)$, set $r_0 = \min\limits_{x \in \partial G} r_0(x)$, and assume that

$$h < \frac{r_0}{10}\,. \tag{2.17}$$

Let $kh \in \partial G_h^+$. The point $\ell h \in \partial G_h^-$ is called the closest to $kh$ if $\operatorname{dist}(\ell h, kh) = h$. The following lemma holds.

**Lemma 2.4.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^\infty$-boundary $\partial G$, and let $h$ satisfy* (2.17)*. Then, if $d = 2$, each point $kh \in \partial G_h^+$ has one or two* (*not more*) *closest points $\ell h \in \partial G_h^-$* (*as illustrated in Fig.* 2.2.) *If $d = 3$, each point $kh \in \partial G_h^+$ has one, two, or three closest points $\ell h \in \partial G_h^-$.*

We have to make more precise what is needed to ensure that the relation (2.14) is valid for every $\{\phi_k, kh \in G_h \cup \partial G_h^+\}$. For this relation, (2.15) should be true for every $\{\phi_k, kh \in \partial G_h^+\}$, i.e., for each $kh \in \partial G_h^+$ the coefficient before $\overline{\phi}_k$ in (2.15) should be equal to zero. First, we calculate these coefficients and write down the boundary conditions for the $d = 2$ case.

(i) *$kh \in \partial G_h^+$ possesses only one closest point from $\partial G_h^-$.* Then either $kh \in \partial G_h^+(+j)$ or $kh \in \partial G_h^+(-j)$ for some $j$. In the first case, $V^j_{k-e_j} = 0$ and, by virtue of (2.16),

$$\psi_k = \psi_{k-e_j}(1 + ihA^j_{k-e_j}), \quad kh \in \partial G_h^+(+j) \quad \text{with } j = 1, 2\,. \tag{2.18}$$

In the second case, $V_k^j(i + hA_k^j) = 0$ and therefore

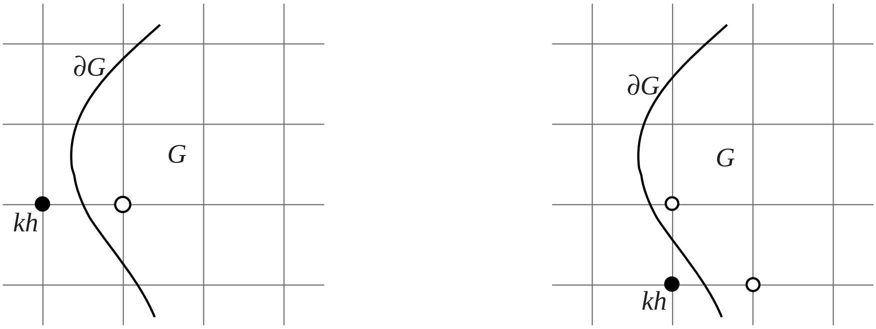


**Fig. 2.2** In the figure on the left, the point $kh \in \partial G_h^+$ (filled circle) has one closest point $\ell h \in \partial G_h^-$ (open circle). In the figure on the right, the point $kh \in \partial G_h^+$ has two closest points $\ell h \in \partial G_h^-$.

$$\psi_k = \frac{\psi_{k+e_j}}{1+ihA_k^j}, \quad kh \in \partial G_h^+(-j) \quad \text{with } j=1,2. \tag{2.19}$$

(ii) *$kh \in \partial G_h^+$ possesses two closest points from $\partial G_h^-$.* Then three different cases are possible.

(1) $kh \in \partial G_h^+(+1) \cap \partial G_h^+(+2)$. In this case, $V_{k-e_1}^1 + V_{k-e_2}^2 = 0$ and, by (2.16),

$$2\psi_k = \left(1+ihA_{k-e_1}^1\right)\psi_{k-e_1} + (1+ihA_{k-e_2}^2)\psi_{k-e_2}. \tag{2.20}$$

(2) $kh \in \partial G_h^+(-1) \cap \partial G_h^+(-2)$. In this case, $V_k^1(i+hA_k^1)+V_k^2(i+hA_k^2)=0$ and

$$\psi_k = \frac{\psi_{k+e_1}(1-ihA_k^1)+\psi_{k+e_2}(1-ihA_k^2)}{2+h^2\left((A_k^1)^2+(A_k^2)^2\right)}. \tag{2.21}$$

(3) $kh \in \partial G_h^+(-j) \cap \partial G_h^+(+\ell)$ for $1 \leqslant j,\ell \leqslant 2$, $\ell \neq j$. In this case, $V_k^j(1-ihA_k^j) - V_{k-e_\ell}^\ell = 0$ and

$$\psi_k = \frac{\psi_{k+e_j}(1-ihA_k^j)+\psi_{k-e_\ell}(1+ihA_{k-e_j}^\ell)}{2+h^2(A_k^j)^2}. \tag{2.22}$$

For the $d=3$ case, the derivation of the boundary conditions is absolutely the same, but the number of distinct cases is larger. Note that, for our purposes, we need only two things from the boundary conditions. First, that the formula (2.14) holds and second, that for each $kh \in \partial G_h^+$, $\psi_k$ is expressed in terms of $\psi_\ell$, $\ell h \in \partial G_h^-$. That is why it is quite enough for us to write down boundary conditions for both the $d=2$ and $d=3$ cases as follows. We have

$$\psi_k = \sum_{j=1}^{3} \left( a_{k,j}^{+} \psi_{k+e_j} + a_{k,j}^{-} \psi_{k-e_j} \right) \quad \forall\, kh \in \partial G_h^{+}\,, \tag{2.23}$$

where $a_{k,j}^{\pm}$ are certain coefficients (that can be written down explicitly) such that if $a_{k,j}^{+} \neq 0$, ($a_{k,j}^{-} \neq 0$) then $h(k+e_j) \in \partial G_h^{-}$ (correspondingly $h(k-e_j) \in \partial G_h^{-}$). Moreover, $0 < \sum\limits_{j=1}^{3} |a_{k,j}^{+}|^2 + |a_{k,j}^{-}|^2 < c$, where $c$ does not depend on $h$.

## 3 The stochastic Ginzburg–Landau Equation

In this section, we provide the formal definition of the Wiener process, the Wiener measure, and some related concepts. Then these results are used in the formulation of the stochastic problem for the Ginzburg–Landau equation.

### *3.1 Wiener process*

We have an abstract probability space $(\Omega, \Sigma, m(d\omega))$, where $\Omega$ is the set of elementary events; $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$ (if $\Omega$ is a metric space, $\Sigma$ is a Borel $\sigma$-algebra, i.e., $\Sigma = \mathcal{B}(\Omega)$ is the $\sigma$-algebra generated by all open subsets of $\Omega$); and $m(d\omega)$ is a probability measure defined on $\Sigma$. Recall that a set $A$ is of $m$-measure zero if there exists $B \in \Sigma$ such that $m(B) = 0$ and $A \subset B$. The $\sigma$-algebra $\Sigma^m$ is called the completion of $\Sigma$ with respect to $m$ if $\Sigma^m$ is the family of all subsets of the form $A \cup B$, where $A$ is of $m$-measure zero and $B \in \Sigma$. In the sequel, we change $\Sigma$ on $\Sigma^m$, i.e., we will consider the $\sigma$-algebra $\Sigma$ that is complete with respect to $m$.

Let

$$W : \Omega \to C(0,\infty; L^2(G)) \equiv \mathcal{C}$$

be a measurable mapping, i.e., for all $B \in \mathcal{B}(\mathcal{C})$, $\{\omega : W(\cdot,\cdot,\omega) \in B\} \in \Sigma$. The probability distribution of $W$ is the measure $\Lambda$ defined on $\mathcal{B}(\mathcal{C})$ by the formula

$$\Lambda(B) \equiv m\big(\{\omega \in \Omega : W(\cdot,\cdot,\omega) \in B\}\big) \quad \forall\, B \in \mathcal{B}(\mathcal{C})\,. \tag{3.1}$$

$W(t,x,\omega)$ is called a Wiener process if $\Lambda(B)$ is a Wiener measure. In the following definition, we assume that $\mathcal{C}$ consists of real-valued functions.

**Definition 3.1.** $\Lambda(B)$ for $B \in \mathcal{B}(\mathcal{C})$ is called a Wiener measure if its Fourier transform $\widetilde{\Lambda}$ is of the form

$$\widetilde{\Lambda}(v) = \int e^{i[w,v]} \Lambda(dW) = e^{-\frac{1}{2}B(v,v)} \quad \forall\, v \in \mathcal{C}_0^{\infty} \equiv C_0^{\infty}((0,\infty)\times G)\,, \tag{3.2}$$

where

$$[w,v]=\int_0^\infty\int_G w(t,x)v(t,x)\,dxdt\,. \tag{3.3}$$

Here, $B(v,v)$ is the quadratic form

$$B(v,v)=\int_0^\infty\int_0^\infty t\wedge s\Big\langle\mathcal{K}\big(v(t,\cdot),v(s,\cdot)\big)\Big\rangle\,dtds\,, \tag{3.4}$$

where $t\wedge s=\min(t,s)$ and $\langle f,g\rangle=\int_G f(x)g(x)\,dx$. Here, $\mathcal{K}$ is a self-adjoint, nonnegative trace class operator in $L^2(G)$ called the correlation operator of $\Lambda$; we have

$$\mathcal{K}^*=\mathcal{K}\geqslant 0,\quad S=S_P\mathcal{K}=\sum_{j=1}^\infty\lambda_j<\infty\quad(S_P\text{ is the spur-trace})\,, \tag{3.5}$$

where $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k\geqslant\cdots\geqslant 0$ is the set of all eigenvalues of the operator $\mathcal{K}$.

Evidently, (3.1)–(3.4) imply that

$$\int W(t,x,\omega)W(s,y,\omega)m(d\omega)=t\wedge s\;\mathcal{K}(x,y)\,, \tag{3.6}$$

where $W(t,x,\omega)$ is a Wiener process and $\mathcal{K}(x,y)$ is the kernel of the operator $\mathcal{K}$ from (3.4) and (3.5).

**Lemma 3.2.** *The following conditions hold.*

1. *For any operator $\mathcal{K}:L^2(G)\to L^2(G)$ satisfying* (3.5) *there exists a unique Wiener measure $\Lambda$ on $\mathcal{C}$ with the correlation operator $\mathcal{K}$.*

2. *For any $\phi,\psi\in L^2(G)$*

$$\int_{\mathbb{C}}\langle W(t,\cdot)\phi(\cdot)\rangle\langle W(s,\cdot)\psi(\cdot)\rangle\Lambda(dW)=t\wedge s\langle\mathcal{K}\phi,\psi\rangle. \tag{3.7}$$

3. *Let $S=S_P\mathcal{K}$ be defined by* (3.5). *Then*

$$\int\|W(t,\cdot)\|^2_{L^2(G)}\Lambda(dW)=tS\quad\forall\,t>0\,. \tag{3.8}$$

4. *$W(t,x,\omega)$ is a process with independent increments, i.e., for any $0\leqslant\tau\leqslant s\leqslant t$,*

$$\begin{aligned}&\Lambda(\{W : W(t,\cdot) - W(s,\cdot) \in B_1, W(\tau,\cdot) \in B_2\})\\ &\quad = \Lambda(\{W : W(t,\cdot) - W(s,\cdot) \in B_1\})\Lambda(\{W : W(\tau,\cdot) \in B_2\})\\ &\qquad\qquad \forall\, B_1, B_2 \in \mathcal{B}(L^2(G)) .\end{aligned} \tag{3.9}$$

For the proof, see [18].

Recall that, given a Wiener measure $\Lambda(B)$, $B \in \mathcal{B}(\mathcal{C})$, one can easily construct a Wiener process for which $\Lambda(B)$ is a probability distribution. Indeed, we take the probability space $(\Omega, \Sigma, m(dW)) = (\mathcal{C}, \mathcal{B}(\mathbb{C}), \Lambda(dW))$ and define a Wiener process $W(t,x,\omega)$ as follows: for each $W \in \mathcal{C}$, $W(t,x,\omega) = W(t,x)$. Clearly, this map $W(t,x,\omega)$ satisfies the definition of a Wiener process.

Below we use Wiener processes $W(t,x,\omega)$ defined on the space $\mathbb{C} = \mathcal{C} + i\mathcal{C}$ of complex valued functions, where recall that $\mathcal{C} = C(0,\infty; L^2(G))$. Taking into account [19, Chapt. III, Sect. 1], we give the following definition.

**Definition 3.3.** The random process $W(t,x,\omega)$, $t \geqslant 0$, $x \in G$, $\omega \in \Omega$, is called a complex Wiener process if

$$W(t,x,\omega) = \mathrm{Re}\, W(t,x,\omega) + i \mathrm{Im}\, W(t,x,\omega) , \tag{3.10}$$

where $\mathrm{Re}\, W(t,x,\omega)$ and $\mathrm{Im}\, W(t,x,\omega)$ are real-valued Wiener processes on $(\Omega, \Sigma, m(d\omega))$ and $W(t,x)$ satisfies the equality

$$\int W(t,x,\omega)W(s,y,\omega)m(d\omega) \equiv 0 \quad \forall\, t \geqslant 0,\ s \geqslant 0, \quad \text{a.e. } x,y \in G . \tag{3.11}$$

It is clear that (3.11) is equivalent to the conditions

$$\begin{aligned} t \wedge s\, \mathcal{K}_{11}(x,y) &\equiv \int \mathrm{Re}\, W(t,x,\omega)\mathrm{Re}\, W(s,y,\omega)m(d\omega)\\ &= \int \mathrm{Im}\, W(t,x,\omega)\mathrm{Im}\, W(s,y,\omega)m(dw)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned} t \wedge s\, \mathcal{K}_{12}(x,y) &\equiv \int \mathrm{Re}\, W(t,x,\omega)\mathrm{Im}\, W(s,y,\omega)m(d\omega)\\ &= -\int \mathrm{Im}\, W(t,x,\omega)\mathrm{Re}\, W(s,y,\omega)m(dw) ,\end{aligned} \tag{3.13}$$

where the first identities in (3.12) and (3.13) are the definitions of $\mathcal{K}_{11}(x,y)$ and $\mathcal{K}_{12}(x,y)$ respectively. By virtue of (3.13), $\mathcal{K}_{12}(x,x) \equiv 0$ and therefore the Wiener processes $\mathrm{Re}\, W(t,x)$ and $\mathrm{Im}\, W(t,x)$ are independent. Moreover, (3.11) implies that

$$t \wedge s\, \mathcal{K}(x,y) \equiv \int W(t,x,\omega)\overline{W(s,y,\omega)}m(d\omega)$$

$$= 2t \wedge s\Big(\mathcal{K}_{11}(x,y) - i\mathcal{K}_{12}(x,y)\Big)\,, \tag{3.14}$$

where the first identity is the definition of $\mathcal{K}(x,y)$. The function $\mathcal{K}(x,y)$ is a non-negative definite kernel; this means that

$$\int\limits_G \int\limits_G \mathcal{K}(x,y)z(y)\overline{z(x)}\,dxdy \geqslant 0 \quad \forall\, z(x) \in L^2(G)\,. \tag{3.15}$$

Here, $z(x)$ is a complex-valued function. As in the real-valued case, we suppose that the operator $\mathcal{K}z = \int\limits_G \mathcal{K}(x,y)z(y)\,dy$ is not only non-negative self-adjoint, but is a trace class operator in $L^2(G)$, i.e.

$$\int\limits_G \mathcal{K}(x,x)\,dx < \infty. \tag{3.16}$$

Moreover, we assume that the kernel $\mathcal{K}$ satisfies the inequality:

$$\int\limits_G \Big(\sum_{j=1}^{d} \frac{\partial^2 \mathcal{K}(x,y)}{\partial_{x_j}\partial_{y_j}}\Big)\Big|_{y=x}\,dx < \infty\,. \tag{3.17}$$

Finally, we denote by $\Lambda(\mathbb{B})$, $\mathbb{B} \in \mathcal{B}(\mathbb{C})$, the Wiener measure, i.e., the distribution of a complex Wiener process $W(t,x)$ from (3.10) and by $\Lambda_R(B_R)$, $B_R \in \mathcal{B}(\mathcal{C})$, and $\Lambda_I(B_I)$, $B_I \in \mathcal{B}(\mathcal{C})$, we respectively denote the Wiener measures of the Wiener processes $\operatorname{Re} W(t,x)$ and $\operatorname{Im} W(t,x)$. It was mentioned above that the Wiener processes $\operatorname{Re} W$ and $\operatorname{Im} W$ are independent. Therefore

$$\Lambda(\mathbb{B}) = \Lambda_R(B_R)\Lambda_I(B_I) \quad \forall\, \mathbb{B} = B_R + iB_I, \quad B_R, B_I \in \mathcal{B}(\mathbb{C})\,. \tag{3.18}$$

## *3.2 The stochastic problem for the Ginzburg–Landau equation*

Let $r(\lambda)$ be the function $\max\{\rho_1, \rho_2|\lambda|\}$, $\lambda \in \mathbb{R}^1$, smoothed in a neighborhood of the points $\lambda = \pm\rho_1/\rho_2$, where $\rho_1 > 0$ and $\rho_2 \geqslant 0$ are given scalars. More precisely, we define $r(\lambda)$ as

$$\begin{cases} r(\lambda) \in C^2(\mathbb{R}^1), \quad r(\lambda) = r(|\lambda|), \\[2ex] r'(\lambda) > 0 \text{ for } \lambda > \dfrac{\rho_1}{2\rho_2}, \quad r''(\lambda) > 0 \text{ for } \dfrac{\rho_1}{2\rho_2} < \lambda < \dfrac{3\rho_1}{2\rho_2}, \\[2ex] r(\lambda) = \max\{\rho_1, \rho_2|\lambda|\} \text{ for } \lambda \in \mathbb{R}^1 \setminus \left\{\dfrac{\rho_1}{2\rho_2} < |\lambda| < \dfrac{3\rho_1}{2\rho_2}\right\}\,. \end{cases} \tag{3.19}$$

For each real-valued function $f(\lambda)$, $\lambda \in \mathbb{R}^1$, and complex number $\psi = \operatorname{Re}\psi + i\operatorname{Im}\psi$, we denote

$$f[\psi] = f(\operatorname{Re}\psi) + if(\operatorname{Im}\psi)\,. \tag{3.20}$$

Moreover, we set, for each complex $z = \operatorname{Re}z + i\operatorname{Im}z$,

$$\widehat{f}[\psi]z = f(\operatorname{Re}\psi)\operatorname{Re}z + if(\operatorname{Im}\psi)\operatorname{Im}z\,. \tag{3.21}$$

This notation will be used throughout the paper. Using this notation, the stochastic Ginzburg–Landau equation we consider has the form

$$d\psi(t,x) + (i\nabla + A)^2\psi - \psi + |\psi|^2\psi = \widehat{r}[\psi(t,x)]dW(t,x)\,, \tag{3.22}$$

where, as in (2.1), $(t,x) \in Q_T \equiv (0,T) \times G$ and the operator $(i\nabla + A)^2$ is defined in (2.4). $W(t,x)$ on the right-hand side of (3.22) is a complex Wiener process introduced in the previous subsection, i.e., $W(t,x) = \operatorname{Re}W(t,x) + \operatorname{Im}W(t,x)$ and $dW(t,x)$ is the corresponding white noise. $r(\cdot)$ is the function defined in (3.19). The solution $\psi(t,x)$ of (3.22) is a complex-valued random function defined on the same probability space $(\Omega, \Sigma, m)$ in which the Wiener process $W(t,x) \equiv W(t,x,\omega)$, $\omega \in \Omega$, is defined, i.e.,

$$\psi(t,x) = \operatorname{Re}\psi(t,x,\omega) + i\operatorname{Im}\psi(t,x,\omega), \quad \omega \in \Omega,$$

is a $\Sigma$-measurable function with respect to $\omega$.

Note that we interpret the right-hand side $\widehat{r}(\psi)dW$ of (3.22) in the sense of (3.21), i.e.,

$$\begin{aligned}&\widehat{r}\big[\psi(t,x)\big]\,dW(t,x)\\ &\quad = r\big(\operatorname{Re}\psi(t,x)\big)d\operatorname{Re}W(t,x) + ir\big(\operatorname{Im}W(t,x)\big)d\operatorname{Im}W(t,x)\,.\end{aligned} \tag{3.23}$$

Each component of the random force should be proportional to the corresponding component of the solution. We introduce $\rho_1$ in the definition of $r(\lambda)$ given in (3.19) because, should the solution be sufficiently small, the consideration of additive white noise as a random force is more natural. Formally, the function defined in (3.19) multiplying the white noise $dW$ allows us to consider the case of additive white noise (when $\rho_1 > 0$, $\rho_2 = 0$) and multiplicative white noise (when $\rho_1 > 0$, $\rho_2 > 0$). However, note that the majority of the difficulties we are forced to overcome are connected with multiplicative white noise.

Equation (3.22) is supplied with the boundary condition (2.2) and the initial condition (2.3). In this case, the initial function $\psi_0(x) = \psi_0(x,\omega)$, $\omega \in \Omega$, is a random function, defined on the same probability space as the Wiener process $W(t,x)$, that has values in $L^1(G)$; $\psi_0 : \Omega \to L^1(G)$. Moreover, we assume that $\psi_0(x,\omega)$ and $W(t,x,\omega)$ are independent.

Finally, note that Equation (3.22) is understood as an Ito differential equation. This means that, by definition, (3.22) is equivalent to the equation

$$\psi(t,x) + \int_0^t \Big[(i\nabla + A)^2\psi(s,x) - \psi(s,x) + |\psi|^2\psi(s,x)\Big]\,ds = \int_0^t \widehat{r}\big[\psi(s,x)\big]\,dW(s,x) + \psi_0(x)\,. \tag{3.24}$$

A more precise definition of the stochastic integral on the right-hand side of (3.24) will be given later.

# 4 Discrete Approximation of the Stochastic Problem

To prove the main result about the existence of a solution for the stochastic Ginzburg–Landau problem, we approximate this problem by the method of lines. In this section, we study these approximations. We begin with the approximation of the Wiener process defined in Sect. 3. For this we need some preliminaries.

## *4.1 Definition of a projector $P_h$ in $L^2(G)$*

For each point $kh \in G_h^0$, $k = (k_1, \dots, k_d)$, we define

$$Q_k = \{x = (x_1, \dots, x_d) \in G \,:\, h(k_j - \frac{1}{2}) \leqslant x_j < h(k_j + \frac{1}{2}),\ \ j = 1, \dots, d\}\,. \tag{4.1}$$

If $kh \in \partial G_h^-(-m)$ and $kh \neq \partial G_h^-(\pm n)$ for each $n \neq m$, we set

$$\begin{aligned} Q_k = \{&x = (x_1, \dots, x_d) \in G \,: \\ &x_m \in [h(k_m - \tfrac{1}{2}), h(k_m + 1)),\ x_j \in [h(k_m - \tfrac{1}{2}), h(k_m + \tfrac{1}{2})), \forall j \neq m\}\,. \end{aligned} \tag{4.2}$$

Analogously, for $kh \in \partial G_h^-(+m)$ such that $kh \neq \partial G_h^-(\pm n)$ for all $n \neq m$, we set

$$\begin{aligned} Q_k = \{&x = (x_1, \dots, x_d) \in G \,: \\ &x_m \in [h(k_m - 1), h(k_m + \tfrac{1}{2})),\ x_j \in [h(k_m - \tfrac{1}{2}), h(k_m + \tfrac{1}{2})), \forall j \neq m\}\,. \end{aligned} \tag{4.3}$$

*Remark* 4.1. We note that the change from (4.1) to (4.2) consists of increasing the interval $x_m \in [h(k_m - \frac{1}{2}), h(k_m + \frac{1}{2}))$ from the right and, in (4.3), this interval is increased from the left.

For each $kh \in \partial G_h^-(-m) \cap \partial G_h^-(-n)$, $kh \neq \partial G_h^-(\pm p)$, if $p \neq n$, $p \neq m$, we define

$$Q_k = \{x = (x_1, \ldots, x_d) \in G \;: \\ x_j \in [h(k_j - \tfrac{1}{2}), h(k_j + 1)), j = n, m;\; x_p \in [h(k_p - \tfrac{1}{2}), h(k_p + \tfrac{1}{2}))\}\,. \tag{4.4}$$

The sets $Q_k$ for $kh \in \partial G_h^-(+m) \cap \partial G_h^-(\pm n)$, $kh \neq \partial G_h^-(\pm p)$ for $p \neq n$, $p \neq m$, and for $kh \in \partial G_h^-(-m) \cap \partial G_h^-(+n)$, $kh \neq \partial G_h^-(\pm p)$, $p \neq n$, $p \neq m$, are defined analogously to (4.4), but with the changes noted in Remark 4.1.

Finally, if $d = 3$, then for each $kh \in \partial G_h^-(-m) \cap \partial G_h^-(-n) \cap \partial G_h^-(-p)$, we set

$$Q_k = \{x = (x_1, \ldots, x_3) \in G \;:\; x_j \in (h(k_j - \frac{1}{2}), h(k_j + 1)), j = 1, 2, 3\}\,. \tag{4.5}$$

In the other cases when $kh \in \partial G_h^-(\pm m) \cap \partial G_h^-(\pm n) \cap \partial G_h^-(\pm p)$, the set $Q_k$ is defined analogously by taking into account Remark 4.1. Important properties of the sets $Q_k$ defined in (4.1)–(4.5) are as follows:

a. for each $k, \ell \in \mathbb{Z}^p$ such that $kh \in G_h$, $\ell h \in G_h$, and $k \neq \ell$, the relation $Q_k \cap Q_\ell = \varnothing$ is true;

b. $\bigcup\limits_{kh \in G_h} Q_k = G.$

For each set $Q_k$ defined in (4.1)–(4.5) we put

$$V(Q_k) = \int\limits_{Q_k} dx\,.$$

Clearly, $V(Q_k) = h^d$ for $Q_k$ defined in (4.1) and, if $h$ is small enough, which is the situation we consider, then

$$\frac{h^d}{4} \leqslant V(Q_k) \leqslant \left(\frac{3}{2}\right)^2 h^d \quad \text{for } Q_k \text{ defined by (4.2)–(4.4)} \tag{4.6}$$

and

$$\frac{h^d}{8} \leqslant V(Q_k) \leqslant \left(\frac{3}{2}\right)^3 h^3 \quad \text{for } Q_k \text{ defined by (4.5).} \tag{4.7}$$

The space $L^{2,h} \equiv L^{2,h}(G_h)$ is defined as the set of lattice functions $\mathbf{f} = \{f_k, kh \in G_h\}$ supplied with the scalar product and norm given by

$$(\mathbf{f}, \mathbf{g})_{L^{2,h}} = h^d \sum_{kh \in G_h} f_k \overline{g_k} \qquad \text{and} \qquad \|\mathbf{f}\|^2_{L^{2,h}} = h^d \sum_{kh \in G_h} |f_k|^2\,, \tag{4.8}$$

respectively. We introduce the operator $P_h$ as follows:

$$P_h \;:\; L^2(G) \to L^{2,h}(G_h) \;\text{ such that }\; (P_h f)_k = V^{-1}(Q_k) \int\limits_{Q_k} f(x)\, dx\,. \tag{4.9}$$

Then, taking into account (4.6) and (4.7), we obtain

$$\begin{aligned}\|P_h f\|^2_{L^{2,h}} &= h^d \sum_{kh\in G_h} V^{-2}(Q_k)\left|\int_{Q_k} f(x)\,dx\right|^2 \\ &\leqslant h^d \sum_{kh\in G_h} V^{-1}(Q_k)\int_{Q_k} |f(x)|^2\,dx \\ &\leqslant 8\int_G |f(x)|^2\,dx = 8\|f\|^2_{L^2(G)}\,. \end{aligned} \tag{4.10}$$

## 4.2 Approximation of Wiener processes

Now let $(\Omega,\Sigma,m)$ be the probability space $\Omega \ni \omega \to W(t,x,\omega) \in \mathbb{C}$, where $W(t,x,\omega)$ is the complex-valued Wiener process introduced in Sect. 3.1; recall that we defined $\mathbb{C} \equiv C(0,\infty;L^2(G))$. In a similar manner, we let $\mathbb{C}_h$ denote $\mathbb{C}_h = C(0,\infty;L^{2,h}(G_h))$. Then the operator (4.9) defines the operator $P_h : \mathbb{C} \to \mathbb{C}_h$. Using this operator, we introduce the projection of the Wiener process on the space $\mathbb{C}_h$ as follows:

$$\mathbf{W}(t,\omega) \equiv \{W_k(t,\omega), kh \in G_h\} = P_h W(t,\cdot,\omega)\,, \tag{4.11}$$

where $W(t,\cdot,\omega) = W(t,x,\omega)$ is the initial Wiener process. We will show that $W_k(t,\omega)$ is a scalar Wiener process and $\mathbf{W}(t,\omega)$ is a vector-valued Wiener process by calculating their probability distributions. Let $\Lambda$ be the distribution defined by (3.1). Recall that, by definition (see [44]),[2]

$$P_h^T \Lambda(\mathcal{B}_h) \equiv P_h^* \Lambda(\mathcal{B}_h) = \Lambda(P_h^{-1}\mathcal{B}_h) \quad \forall\, \mathcal{B}_h \in \mathcal{B}(\mathbb{C}_h)\,, \tag{4.12}$$

where $P_h^{-1}\mathcal{B}_h = \{\omega \in \mathbb{C} : P_h\omega \in \mathcal{B}_h\}$. This definition is equivalent to the expression

$$\int_{\mathbb{C}_h} F(\mathbf{W}) P_h^* \Lambda(d\mathbf{W}) = \int_{\mathbb{C}} F(P_h \mathrm{W})\Lambda(d\mathrm{W}) = \int_{\Omega} F(P_h \mathrm{W}(\cdot,\omega)) m(d\omega) \tag{4.13}$$

for every $F$ for which at least one integral from (4.13) is well-defined. Note that the operator $P_h^* : L^{2,h}(G_h) \to L^2(G)$ is the adjoint of the operator (4.9) and is defined as

[2] In addition to the standard notation $P_h^*\Lambda$, we also introduce $P_h^T\Lambda$ in order to avoid confusion in (4.15).

$$(P_h^*\mathbf{f})(x) = f_h(x) = \sum_{kh\in G_h} f_k h^d V^{-1}(Q_k)\mathcal{X}_{Q_k}(x), \quad x \in G\,, \tag{4.14}$$

where $\mathbf{f} = \{f_k\} \in L^{2,h}(G_h)$ and $\mathcal{X}_{Q_k}(x)$ is the characteristic function of the set $Q_k$, i.e., $\mathcal{X}_{Q_k}(x) = 1$ for $x \in Q_k$ and $\mathcal{X}_{Q_k}(x) = 0$ for $x \neq Q_k$.

Taking into account (3.2), (4.13), and (4.14), we have

$$\begin{aligned}\widetilde{P_h^T \Lambda}(\mathbf{v}) \equiv \widetilde{P_h^* \Lambda}(\mathbf{v}) &= \int_{\mathbb{C}_h} e^{i\int_0^\infty (\mathbf{W}(t),\mathbf{v}(t))_{L^{2,h}}\,dt} P_h^*\Lambda(dW)\\ &= \int_{\mathbb{C}_h} e^{i\int_0^\infty (P_h\mathbf{W})(t),\mathbf{v}(t))_{L^{2,h}}\,dt}\Lambda(dW) = \int_{\mathbb{C}} e^{i[\mathrm{W},P_h^*\mathbf{v}]}\Lambda(dW)\\ &= e^{-\frac{1}{2}B(P_h^*\mathbf{v},P_h^*\mathbf{v})}\,.\end{aligned} \tag{4.15}$$

By virtue of (3.4) and (4.14),

$$B_h(\mathbf{v},\mathbf{v}) \equiv B(P_h^*\mathbf{v}, P_h^*\mathbf{v}) = \int_0^\infty\int_0^\infty t\wedge s\; h^{2d} \sum_{\substack{\mathbf{j}h\in G_h\\ kh\in G_h}} \mathcal{K}_{jk} v_k(t)\overline{v_j(s)}\;dtds\,, \tag{4.16}$$

where

$$\mathcal{K}_{jk} = V^{-1}(Q_j)V^{-1}(Q_k)\int_{Q_j}\int_{Q_k} \mathcal{K}(x,y)\mathcal{X}_{Q_j}(x)\mathcal{X}_{Q_k}(y)\;dxdy \tag{4.17}$$

and $\mathcal{K}(x,y)$ is the kernel defined in (3.14). The corresponding correlation operator $\mathcal{K}$ is defined by the equality

$$\int_G\int_G \mathcal{K}(x,y)u(y)\overline{v}(x)\;dxdy = (\mathcal{K}u,v)_{L^2(G)} \quad \forall\, u,v \in L^2(G)\,. \tag{4.18}$$

The equality $\mathcal{K} = \mathcal{K}^*$ implies that $\mathcal{K}(x,y) = \overline{\mathcal{K}(y,x)}$ and therefore $\mathcal{K}_{jk} = \overline{\mathcal{K}}_{kj}$. Formulas (4.15)–(4.18) show that $\mathbf{W}(t,\omega)$ is defined. In other words, the matrix $\widehat{\mathcal{K}} = \|h^d\mathcal{K}_{ij}\|$ is reduced to diagonal form by the unitary transformation $\widehat{\Theta} = \|\Theta_{ij}\|$, i.e.,

$$\widehat{\Theta}^*\widehat{\mathcal{K}}\widehat{\Theta} = \widehat{L}\,, \qquad \text{where } \widehat{L} = \|\widehat{L}_{ik}\| = \|\delta_{jk}\mu_k\|\,. \tag{4.19}$$

Here, $\mu_k$ are the eigenvalues of the operator $\widehat{\mathcal{K}} = \|h^d\mathcal{K}_{ij}\|$. Since (3.15) implies the positive semidefiniteness of $\widehat{\mathcal{K}}$, the inequalities $\mu_k \geqslant 0$ hold.

**Lemma 4.1.** *The bound*

$$\sum_{jh\in G_h} \mu_j \leqslant C \int\limits_G \mathcal{K}(x,x)\, dx < \infty \tag{4.20}$$

*holds, where* $\mathcal{K}(x,y)$ *is the kernel* (3.14) *and* $C>0$ *does not depend on* $h$*.*

*Proof.* By virtue of (4.17),

$$\sum_j \mu_j = h^d \sum_j \mathcal{K}_{jj} = h^d \sum_j V^{-2}(Q_j) \int\limits_{Q_j}\int\limits_{Q_j} \mathcal{K}(x,y)\mathcal{X}_{Q_j}(x)\mathcal{X}_{Q_j}(y)\, dxdy\,. \tag{4.21}$$

It is well known that

$$\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}\,, \tag{4.22}$$

where $e_j(x), \lambda_j$ are the eigenfunctions and eigenvalues corresponding to $\mathcal{K}(x,y)$. From this equality we have

$$\begin{aligned} |\mathcal{K}(x,y)| &\leqslant \sum_{j=1}^{\infty} \lambda_j |e_j(x)|\, |e_j(y)| \\ &\leqslant \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j \Big(|e_j(x)|^2 + |e_j(y)|^2\Big) = \frac{1}{2}\Big(\mathcal{K}(x,x) + \mathcal{K}(y,y)\Big)\,. \end{aligned} \tag{4.23}$$

Substituting this inequality into (4.21), we find

$$\begin{aligned} \sum_j \mu_j &\leqslant h^d \sum_j V^{-1}(Q_j)\frac{1}{2}\Big(\int\limits_{Q_j} \mathcal{K}(x,x)\mathcal{X}_j(x)\, dx + \int\limits_{Q_j} \mathcal{K}(y,y)\mathcal{X}_j(y)\, dy\Big) \\ &\leqslant C\int\limits_G \mathcal{K}(x,x)\, dx < \infty\,. \end{aligned} \tag{4.24}$$

The lemma is proved □

We set

$$\widetilde{\mathbf{v}}(t) = \varTheta^* \mathbf{v}(t) \qquad \text{and} \qquad \widetilde{\mathbf{W}}(t,\omega) = \varTheta^* \mathbf{W}(t,\omega)\,. \tag{4.25}$$

Since $\varTheta^* = \varTheta^{-1}$, we have, by (4.15), (4.16), and (4.19), that

$$\begin{aligned} \widetilde{P_h^* \varLambda}(\mathbf{v}) &= e^{-\frac{1}{2}\int_0^\infty\int_0^\infty t\wedge s(\widehat{\mathcal{K}}\mathbf{v}(t),\mathbf{v}(s))_{L^{2,h}}\, dtds} \\ &= e^{-\frac{1}{2}\int_0^\infty\int_0^\infty t\wedge s(\widehat{\mathcal{K}}\varTheta\widetilde{\mathbf{v}}(t),\varTheta\widetilde{\mathbf{v}}(s))_{L^{2,h}}\, dtds} \end{aligned}$$

$$= e^{-\frac{1}{2}\int_0^\infty\int_0^\infty t\wedge s\ h^d \sum_k \mu_k \widetilde{v}_k(t)\overline{\widetilde{v}_k(s)}\ dtds}$$

$$= \prod_k e^{-\frac{h^d}{2}\int_0^\infty\int_0^\infty t\wedge s \mu_k v_k(t)\overline{\widetilde{v}_k(s)}\ dtds}$$

$$= \prod_k \int_\Omega e^{ih^d\int_0^\infty \widetilde{W_k}(t,\omega)\overline{\widetilde{v}_k(t)}dt} m(d\omega)\,.$$

Hence,

$$\int_\Omega e^{i\int_0^\infty (\widetilde{\mathbf{W}}(t,\omega),\widetilde{\mathbf{v}}(t))dt} m(d\omega) = \prod_k \int_\Omega e^{ih^d\int_0^\infty \widetilde{W_k}(t,\omega)\overline{\widetilde{v_k}(t)}dt} m(d\omega)\,. \tag{4.26}$$

This equality implies that the scalar Wiener processes $\widetilde{W_k}(t,\omega)$ for $kh \in G_h$ are independent. For the definition of independence of scalar Wiener processes, see [26, p. 55].

## 4.3 The Ito integral

Together with the probability space $(\Omega, \Sigma, m)$ and the Wiener process $W(t,x)$ introduced in Sect. 3, we consider the increasing filtration $\Sigma_t$ (see [26, p. 52]), i.e., a collection of $\sigma$-fields $\Sigma_t \subset \Sigma$, defined for each $t$, such that $\Sigma_s \subset \Sigma_t$ for $t \geqslant s$. Also, we assume that $W(t,\cdot)$ is $\Sigma_t$-measurable for every $t$ and $W(t+h,\cdot) - W(t,\cdot)$ is independent on $\Sigma_t$. The last statement means that for every $\mathcal{A} \in \Sigma_t$ and $B \in \mathcal{B}(L^2(G))$

$$m\Big(\mathcal{A} \cap \{W(t+h,\cdot) - W(t,\cdot) \in B\}\Big) = m(\mathcal{A})m\Big(\{W(t+h,\cdot) - W(t,\cdot) \in B\}\Big)\,.$$

Then $W(t,x)$ is called the Wiener process relative to the filtration $\Sigma_t$ and the pair $(\mathrm{W}(t,\cdot), \Sigma_t)$ is called a Wiener process.

The operator $P_h$ defined in (4.9) generates the operator $P_h : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^{2,h}(G_h))$ and therefore generates the operator of filtrations

$$P_h : \Sigma_t \to \Sigma_{h,t}\,, \tag{4.27}$$

where, by definition, $B_h \in \Sigma_{h,t}$ if there exists a set $B \in \Sigma_t$ such that $B_h = P_h B$. It is clear that the pair $(\mathbf{W}(t), \Sigma_{h,t})$ is a Wiener process.

Recall (see [26, p. 66]) that a vector-valued function $\mathbf{f}(t,\omega)$ given on $(0,\infty) \times \Omega$ is called $\Sigma_{h,t}$ adapted if it is $\Sigma_{h,t}$-measurable for each $t > 0$. By $\Upsilon$ we denote the set of all $\Sigma_{h,t}$ adapted vector-valued functions which are $\mathcal{B}(0,\infty) \otimes \Sigma_h$ measurable (recall that $\Sigma_h = P_h\Sigma$) and satisfy

$$E\int_0^\infty \mathbf{f}^2(t)\,dt \equiv \int_\Omega \int_0^\infty \mathbf{f}(t,\omega)^2\,dt\,m(d\omega) < \infty\,,$$

where we have used the definition of the mathematical expectation. Here $\mathbf{f} = (f_1, \ldots, f_K)$ where $K$ is the number of points in the grid $kh$ belonging to $G_h$: $K = \#\{k \in \mathbb{Z}^d\ :\ kh \in G_h\}$.

It is well known (see [26, p. 68]) that the Ito integral of a $\Sigma_{h,t}$-adapted function is defined as follows:

$$\int_0^\infty \widehat{\mathbf{f}}(t)d\mathbf{W}(t) = \lim \sum_{j=0}^\infty \mathbf{f}(t_j)(\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\,, \tag{4.28}$$

where $\sup_j |t_{j+1} - t_j| \to 0$ and this limit is understood in the sense of the space $L^2(\Omega, m)$. Here, in accordance with (3.20) and (3.21),

$$\widehat{\mathbf{f}}(t)d\mathbf{W} = \sum_{k=1}^K (\mathrm{Re}\, f_k(t)d\mathrm{Re}\, W_k(t) + i\mathrm{Im} f_k(t)d\mathrm{Im} W_k(t)).$$

By the definition of $\Sigma_{h,t}$-adaptiveness of $\mathbf{f}(t)$, we have, since $E\mathbf{W}(t) = 0$,

$$\begin{aligned} E\int_0^\infty \widehat{\mathbf{f}}(t)d\mathbf{W}(t) &= \lim \sum_{i=0}^\infty E\left(\widehat{\mathbf{f}}(t_j)(\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\right) \\ &= \sum_{i=0}^\infty E\left(\widehat{\mathbf{f}}(t_j)\right) E\left((\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\right) = 0\,. \end{aligned} \tag{4.29}$$

## *4.4 The discrete stochastic system*

We consider the following discrete analogue of the stochastic Ginzburg–Landau equation given in (3.22):

$$\begin{aligned} d\psi_k(t) + \big\{(i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + |\psi_k(t)|^2\psi_k(t)\big\}\,dt \\ = \widehat{r}\big[\psi_k(t)\big]\,dW_k(t)\,, \end{aligned} \tag{4.30}$$

where $W_k(t) = W_k(t,\omega)$ are the scalar Wiener processes introduced in Sect. 4.2, $dW_k(t)$ is white noise, $\boldsymbol{\psi}(t) = \{\psi_k(t), kh \in G_h\}$ is the unknown stochastic vector-valued process that we seek, and $r(\lambda)$ is the function given in (3.19). As was the case for (3.22), the right-hand side of (4.30) is interpreted in accordance with (3.20) and (3.21). If $\psi_k(t) = \mathrm{Re}\,\psi_k(t) + i\mathrm{Im}\,\psi_k(t)$ and $dW_k(t) = d\mathrm{Re}\,W_k(t) + id\mathrm{Im}\,W_k(t)$, then, by definition, we have

$$\widehat{r}\big[\psi_k(t)\big]\,dW_k(t) = r(\mathrm{Re}\,\psi_k(t))d\mathrm{Re}\,W_k(t) + ir(\mathrm{Im}\,\psi_k(t))d\mathrm{Im}\,W_k(t)\,. \tag{4.31}$$

We assume that the solution $\boldsymbol{\psi}(t) = \{\psi_k(t)\}$ of the system (4.30) satisfies the initial condition (2.12) and the boundary condition (2.23).

The problem (4.30), along with (2.12) and (2.23), is the *differential form* of the Ito system that by definition is equivalent to the integral form

$$\begin{aligned} \psi_k(t) = \psi_{0,k} - \int_0^t \left\{ (i\nabla_h + A_k)^2 \psi_k(\tau) - \psi_k(\tau) + |\psi_k(\tau)|^2 \psi_k(\tau) \right\} \, d\tau \\ + \int_0^t \widehat{r}\big[\psi_k(\tau)\big] dW_k(\tau) \quad kh \in G_h \end{aligned} \tag{4.32}$$

combined with the boundary condition (2.23). The Ito integral from (4.32) is defined by (4.28).

## 4.5 The Ito formula

To derive a priori estimates, we use the Ito formula written in convenient form; it is formulated as follows. Let $(\mathbf{W}(t), \Sigma_{h,t})$ be a Wiener process, where $\mathbf{W}(t)$ is defined by (4.11) and $\Sigma_{h,t}$ is defined by (4.27). Suppose that $\boldsymbol{\sigma}(t,\omega)$ is a $K \times K$-matrix-valued function[3] with elements $\boldsymbol{\sigma}_{k,\ell}(t,\omega)$ that are $2 \times 2$ real-valued matrices, i.e., $\boldsymbol{\sigma}_{k,\ell}(t,\omega) = \sigma_{k,\ell,i,j}(t,\omega)$, $i,j = 1,2$. The functions $\sigma_{k,\ell,i,j}(t,\omega)$ are assumed to be $\Sigma_{h,t}$-adapted random functions, $B(0,\infty) \times \Sigma_h$ measurable, and satisfy

$$E \int_0^\infty |\sigma_{k,\ell}(t,\omega)|^2 \, dt \equiv \int_\Omega \int_0^\infty |\sigma_{k,\ell}(t,\omega)|^2 | dt \, m(d\omega) < \infty \, .$$

We set, by definition,

$$\boldsymbol{\sigma}(t) d\{W\}(t) = \Big\{ \sum_{\ell h \in G_h} \widehat{\boldsymbol{\sigma}}_{k,\ell}(t) dW_\ell(t), \quad kh \in G_h \Big\} \, ,$$

where

$$\begin{aligned} \widehat{\boldsymbol{\sigma}}_{k,\ell} dW_\ell = \big(\sigma_{k,\ell,1,1} d\mathrm{Re}\, W_\ell + \sigma_{k,\ell,1,2} d\mathrm{Im}\, W_\ell\big) \\ + i\big(\sigma_{k,\ell,2,1} d\mathrm{Re}\, W_\ell + \sigma_{k,\ell,2,2} d\mathrm{Im}\, W_\ell\big) \, . \end{aligned}$$

---

[3] Recall that $K = \#\{k \in \mathbb{Z}^d \, : \, kh \in G_h\}$.

Let $\mathbf{b}(t,\omega)$ be a $K$-dimensional vector-valued random process (with complex components $b_i(t,\omega)$) that is jointly measurable in $(t,\omega)$, $\Sigma_{h,t}$-adapted, and $\int\limits_0^T |b_s|\, dx < \infty$ a.s.

**Definition 4.2.** A continuous, $\Sigma_{h,t}$-adapted $\mathbb{C}^K$-valued random process $\boldsymbol{\psi}(t,\omega) = (\psi_1, \ldots, \psi_K)$ has the stochastic differential

$$d\boldsymbol{\psi}(t) = \widehat{\boldsymbol{\sigma}}(t) d\mathbf{W}(t) + \mathbf{b}(t)dt \tag{4.33}$$

if and only if, a.s. for all $t$,

$$\boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 + \int\limits_0^t \widehat{\boldsymbol{\sigma}}(s) d\mathbf{W}(s) + \int\limits_0^t \mathbf{b}(s)\, ds\,. \tag{4.34}$$

**Theorem 4.3 (**Ito's formula). *Let $u(x)$ be a real-valued, twice continuously differentiable function of $x \in \mathbb{C}^K$, and let $\boldsymbol{\psi}(t)$ be the random process from Definition* 4.2. *Then $u(\boldsymbol{\psi}(t))$ has a stochastic differential and*

$$\begin{aligned} du(\boldsymbol{\psi}(t)) = \sum_j \frac{\partial u(\boldsymbol{\psi}(t)}{\partial \psi_j} d\psi_j + \frac{\partial u(\boldsymbol{\psi}(t)}{\partial \overline{\psi_j}} d\overline{\psi_j} \\ + \frac{1}{2} \sum_{j,n} \Big( \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \psi_j \partial \psi_n} d\psi_j d\psi_n + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \psi_j \partial \overline{\psi_n}} d\psi_j d\overline{\psi}_n \\ + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \overline{\psi_j} \partial \psi_u} d\overline{\psi_j} d\psi_n + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \overline{\psi_j} \partial \overline{\psi_n}} d\overline{\psi_j} d\overline{\psi_n} \Big)\,. \end{aligned} \tag{4.35}$$

In addition to calculating the products $d\psi_j d\psi_n$, $d\overline{\psi_j} d\psi_n$, $d\psi_j d\overline{\psi_n}$, and $d\overline{\psi_j} d\overline{\psi_n}$, one has to take into account the following rules for calculating products of independent Wiener processes $\widetilde{W}_j(t) \equiv \mathrm{Re}\, \widetilde{W}_j + i \mathrm{Im}\, \widetilde{W}_j$ :

$$\begin{aligned} &d\mathrm{Re}\, \widetilde{W}_j(t) d\mathrm{Re}\, \widetilde{W}_k(t) = d\mathrm{Im}\, \widetilde{W}_j(t) d\mathrm{Im}\, \widetilde{W}_k(t) = \mu_k \delta_{jk} dt \\ &d\mathrm{Re}\, \widetilde{W}_j(t) dt = d\mathrm{Im}\, \widetilde{W}_j(t) dt = 0, \quad d\mathrm{Re}\, \widetilde{W}_j(t) d\mathrm{Im}\, \widetilde{W}_k(t) = 0\,. \end{aligned} \tag{4.36}$$

The proof of the Ito formula is given in [26] for the case of real-valued functions. One can easily reduce the case of complex-valued functions to that of real-valued functions by treating $\mathbb{C}^K$ as $\mathbb{R}^{2K}$.

To derive a priori estimates, we will need the following corollary of (4.36). (Below we will use the definition (4.31).)

**Lemma 4.4.** *The following relationships hold:*

$$\left(\widehat{r}[\psi_k]dW_k\right)^2 = \left(\overline{\widehat{r}[\psi_k]dW_k}\right)^2 = \left(r^2(\operatorname{Re}\psi_k) - r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2 \mu_j \quad (4.37)$$

*and*

$$(\widehat{r}[\psi_k]dW_k)\left(\overline{\widehat{r}(\psi_k)dW_k}\right) = \left(r^2(\operatorname{Re}\psi_k) + r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2 \mu_j\,, \quad (4.38)$$

*where* $\mu_k$ *are the eigenvalues of the correlation operator* (4.17) *for the Wiener process* $\mathbf{W}(t)$ *and* $\Theta_{kj}$ *are elements of the unitary matrix given in* (4.19) *that reduces* (4.17) *to diagonal form.*

*Proof.* We begin from the proof of the following corollaries of (4.36):

$$(d\operatorname{Re} W_k)^2 = (d\operatorname{Im} W_k)^2 = \sum_j |\Theta_{kj}|^2\mu_j dt \quad \text{and} \quad (d\operatorname{Re} W_k)(d\operatorname{Im} W_k) = 0\,. \quad (4.39)$$

By virtue of (4.25), $W_k = \sum_j \Theta_{kj} d\widetilde{W_j}$ and therefore, using (4.36), we obtain

$$\begin{aligned}(d\operatorname{Re} W_k)^2 &= \Big(\sum_j (\operatorname{Re}\Theta_{kj} d\operatorname{Re}\widetilde{W_j} - \operatorname{Im}\Theta_{kj} d\operatorname{Im}\widetilde{W_j})\Big)^2 \\ &= \sum_j \Big((\operatorname{Re}\Theta_{kj})^2 + (\operatorname{Im}\Theta_{kj})^2\Big)\mu_j dt = \sum_j |\Theta_{kj}|^2\mu_j dt\,.\end{aligned}$$

The second and third equalities in (4.39) are proved in the same manner.

By (4.31), (4.36), and (4.39), we have

$$\begin{aligned}\left(\widehat{r}[\psi_k]dW_k\right)^2 =& \Big(r(\operatorname{Re}\psi_k)d\operatorname{Re} W_k + ir(\operatorname{Im}\psi_k)d\operatorname{Im} W_k\Big)^2 \\ =& r^2(\operatorname{Re}\psi_k)(d\operatorname{Re} W_k)^2 - r^2(\operatorname{Im}\psi_k)(d\operatorname{Im} W_k)^2 \\ &+ 2ir(\operatorname{Re}\psi_k)r(\operatorname{Im}\psi_k)d\operatorname{Re} W_k d\operatorname{Im} W_k \\ =& \left(r^2(\operatorname{Re}\psi_k) - r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2\mu_j dt\,.\end{aligned}$$

This equality and the fact that its right-hand side is a real function prove (4.37). The relation (4.38) is proved analogously. □

## 5 A Priori Estimates

In order to prove the solvability not only of the discrete stochastic system (4.30), (2.12), and (2.23), but also of the main stochastic problem (3.22), (2.2), and (2.3), we have to establish a number of a priori estimates for the system (4.30).

## 5.1 Application of the Ito formula

We take the function $u(\boldsymbol{\psi})$ from Theorem 4.3 as

$$u(\boldsymbol{\psi}) = h^d \sum_{hk\in G_h} |\psi_k(t)|^{2p} \equiv \|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} , \tag{5.1}$$

where $p = 1$ or $p = 2$. Applying (5.1) in the Ito formula with the stochastic differential $du$ defined in (4.35) and using (4.37) and (4.38), we obtain

$$\begin{aligned}
d\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} &= h^d \sum_k \Big\{ p|\psi_k|^{2p-2} \left(\overline{\psi}_k d\psi_k + \psi_k d\overline{\psi}_k\right) \\
&\quad + \frac{1}{2} p\cdot(p-1)|\psi_k|^{2(p-2)} \left(\overline{\psi}_k^2 d\psi_k d\psi_k + \psi_k^2 d\overline{\psi}_k d\overline{\psi}_k\right) \\
&\quad + p^2|\psi_k|^{2(p-1)} d\psi_k d\overline{\psi}_k \Big\} \\
&= h^d \sum_k \Big\{ p|\psi_k|^{2p-2} \Big\{ \left(-\overline{\psi}_k (i\nabla + A_k)^2\psi_k + 2|\psi_k|^2 - 2|\psi_k|^4\right) dt \\
&\quad + \overline{\psi}_k \widehat{r}[\psi_k] dW_k - \psi_k \overline{(i\nabla_h + A_k)^2\psi_k} dt \\
&\quad + \psi_k \overline{\widehat{r}[\psi_k] dW_k} \Big\} + \frac{1}{2} p(p-1)|\psi_k|^{2(p-2)} \\
&\quad \Big( \overline{\psi}_k^2 \left\{ \left(-(i\nabla_h + A_k)^2\psi_k + \psi_k - |\psi_k|^2\psi_k\right) dt + \widehat{r}[\psi_k] dW_k \right\}^2 \\
&\quad + \psi_k^2 \left\{ \left(-\overline{(i\nabla_h + A_k)^2\psi_k} + \overline{\psi_k} - |\psi_k|^2 \overline{\psi_k}\right) dt + \overline{\widehat{r}[\psi_k] dW_k} \right\}^2 \Big) \\
&\quad + p^2|\psi_k|^{2(p-1)} \Big\{ \left(-(i\nabla_h + A_k)^2\psi_k + \psi_k - |\psi_k|^2\psi_k\right) dt \\
&\quad + \widehat{r}[\psi_k] dW_k \left\{ \left(-\overline{(i\nabla_h + A_k)^2\psi_k} + \overline{\psi_k} - |\psi_k|^2 \overline{\psi_k}\right) dt + \overline{\widehat{r}[\psi_k] dW_k} \right\} \Big\}
\end{aligned}$$

so that

$$
\begin{aligned}
d\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} & \\
&= h^d \sum_k p|\psi_k|^{2p-2}\Big\{\big(-2\mathrm{Re}\,(\overline{\psi}_k(i\nabla_h + A_k)^2\psi_k) + 2|\psi_k|^2 - 2|\psi_k|^4\big)\,dt \\
&+ 2\mathrm{Re}\,(\overline{\psi}_k\widehat{r}[\psi_k]dW_k)\Big\} \\
&+ h^d\sum_k\Big\{p(p-1)|\psi_k|^{2(p-2)}\mathrm{Re}\,(\psi_k^2)(r^2(\mathrm{Re}\,\psi_k) \\
&- r^2(\mathrm{Im}\,\psi_k))\sum_j|\Theta_{kj}|^2\mu_j \\
&+ p^2|\psi_k|^{2(p-1)}\Big(r^2(\mathrm{Re}\,\psi_k) + r^2(\mathrm{Im}\,\psi_k)]\Big)\sum_j|\Theta_{kj}|^2\mu_j\Big\}dt\,,
\end{aligned}
\tag{5.2}
$$

where $\sum\limits_k = \sum\limits_{kh\in G_h}$. Applying (2.14) with $\overline{\phi}_k = |\psi_k|^{2p-2}\overline{\psi}_k$ to the first term on the right-hand side of (5.2) results in

$$
\begin{aligned}
&-h^d\sum_{kh\in G_h}2p|\psi_k|^{2p-2}\mathrm{Re}\,\left(\overline{\psi}_k(i\nabla_h + A_k)^2\psi_k\right) \\
&\quad = -h^d\widetilde{\sum_{jk}}2p\mathrm{Re}\,\left\{\left((i\partial^+_{j,h} + A^j_k)\psi_k, \overline{(i\partial^+_{j,h} + A^j_k)(|\psi_k|^{2p-2}\psi_k)}\right)\right\}\,,
\end{aligned}
\tag{5.3}
$$

where, for brevity, we use the following notation:

$$
\begin{aligned}
&\widetilde{\sum_{jk}}\left((i\partial^+_{j,h} + A^j_k)\psi_k, \overline{(i\partial^+_{j,h} + A^j_k)\phi_k}\right) \\
&\qquad = \sum_{j=1}^d\sum_{kh\in G_h\cup\partial G^+_h(-j)}\left(i\partial^+_{j,h}\psi_k + A^j_k\psi_k\right)\left(\overline{i\partial^+_{j,h}\phi_k + A^j_k\phi_k}\right).
\end{aligned}
\tag{5.4}
$$

Below, we will also use the notation $\widetilde{\sum\limits_{jk}}$ when in (5.4), $A_k = \{A^j_k\}$ is absent. Moreover, in the next subsection we use the following notation which is closely related to (5.4):

$$
\|\nabla^+_h\psi\|^2_{L^{2,h}} = \widetilde{\sum_{j,k}}|\partial^+_{j,h}\psi_k|^2 \equiv \sum_{j=1}^d\sum_{kh\in G_h\cup\partial G^+_h(-j)}|\partial^+_{j,h}\psi_k|^2\,. \tag{5.5}
$$

## 5.2 A priori estimate for $p = 1$

The following assertion holds.

**Theorem 5.1.** *Let a random process* $\{\boldsymbol{\psi}(t)\} = \{\psi_k\}$ *have the stochastic differential* (4.30)*. Then* $\boldsymbol{\psi}$ *satisfies the estimate*

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + E\int_0^t \left(\|\nabla_h^+\boldsymbol{\psi}(\tau)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(\tau)\|^4_{L^{4,h}}\right) d\tau \leqslant C_2\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right) e^{C_1 t}, \tag{5.6}$$

*where the constants* $C_1$ *and* $C_2$ *do not depend on* $h$*.*

*Proof.* The equality (5.3) for $p = 1$ can be rewritten as follows:

$$-h^d \sum_k 2\mathrm{Re}\,\{\overline{\psi}_k (i\nabla_h + A_k)^2 \psi_k\} = -h^d \widetilde{\sum_{jk}} 2\left|(i\partial^+_{j,h} + A^j_k)\psi_k\right|^2 .$$

Here and in the sequel, we use the notation $\sum\limits_k = \sum\limits_{kh\in G_h}$ as well as the notation (5.4). We substitute this equality into the right-hand side of (5.2) to obtain

$$\begin{aligned} d\|\boldsymbol{\psi}\|^2_{L^{2,h}} &= -2h^d\left[\widetilde{\sum_{jk}}\left|(i\partial^+_{j,h} + A^j_k)\psi_k)\right|^2 - \sum_k\left(|\psi_k|^2 - |\psi_k|^4\right)\right] dt \\ &\quad + 2h^d \mathrm{Re}\sum_k (\overline{\psi}_k \widehat{r}[\psi_k] dW_k) \\ &\quad + h^d\sum_k\Big(r^2(\mathrm{Re}\,\psi_k) + r^2(\mathrm{Im}\,\psi_k)\Big)\sum_j |\Theta_{kj}|^2\mu_j\, dt\,. \end{aligned} \tag{5.7}$$

By virtue of the definition (3.19) for the function $r(\lambda)$ and (3.20), we have

$$|r[\psi_k]|^2 \equiv |r(\mathrm{Re}\,\psi_k)|^2 + |r(\mathrm{Im}\,\psi_k)|^2 \leqslant C^2(1 + |\psi_k(t)|)^2\,. \tag{5.8}$$

An equivalent integral form of the Ito differential is written as

$$\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2\int_0^t h^d\left(\widetilde{\sum_{jk}}|(i\partial^+_{j,h} + A^j_k)\psi_k|^2 + \sum_k|\psi_k|^4\right) d\tau - 2\int_0^t h^d\sum_k \mathrm{Re}\,(\overline{\psi}_k\widehat{r}[\psi_k]dW_k)$$

$$= \int\limits_0^t \left( h^d \sum_k (2|\psi_k|^2 + \sum_j |\Theta_{kj}|^2 \mu_j |r[\psi_k]|^2 \right) d\tau + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.9}$$

Thus, assuming that $\boldsymbol{\psi}(t)$ is a $\Sigma_{h,t}$ adaptive vector function, we apply the mathematical expectation to (5.9). Then, taking into account (5.8), (4.29), the bound $\sum\limits_j |\Theta_{kj}|^2 \mu_j \leqslant \sum\limits_j \mu_j$, and (4.23), we obtain

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2E \int\limits_0^t h^d \left( \widetilde{\sum_{jk}} |(i\partial^+_{j,h} + A^j_k)\psi_k(t)|^2 + \sum_k |\psi_k(t)|^4 \right) d\tau$$

$$\leqslant E \int\limits_0^t C \left( \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 1 \right) d\tau + E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.10}$$

Using the fact that

$$|(i\nabla^+_h + A_k)\psi_k|^2 \geqslant |\nabla^+_h \psi_k|^2 - C|\psi_k|^2 , \tag{5.11}$$

we obtain from (5.10) that

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2E \int\limits_0^t \left( \|\nabla^+_h \boldsymbol{\psi}\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^4_{L^{4,h}} \right) d\tau$$

$$\leqslant C_1 \left( E \int\limits_0^t \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \, d\tau + t \right) + E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.12}$$

Note that the term $|\psi_k|^2$ from (5.11) with $kh \in \partial G^+_h$ can be estimated by $\|\boldsymbol{\psi}\|_{L^{2,h}}$ by virtue of (2.23) and the bounds following that inequality. Now, by applying the Gronwall inequality to (5.12), we finally obtain the desired estimate (5.6). □

## *5.3 A priori estimate for p = 2*

We now establish the following bound.

**Theorem 5.2.** *Let a random process $\boldsymbol{\psi}(t) = \{\psi_k\}$ have the stochastic differential* (4.30). *Then $\boldsymbol{\psi}$ satisfies the estimate*

$$E(\|\boldsymbol{\psi}(t)\|^4_{L^{4,h}} + E \int\limits_0^t \left( \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}} + h^d \widetilde{\sum_{j,k}} |\partial^+_{j,h}\psi_k|^2 |\psi_k|^2 \right) d\tau$$

$$\leqslant C_1 \left(1 + E\|\boldsymbol{\psi}_0\|_{L^4}^4\right) e^{Ct}\,, \tag{5.13}$$

*where $C$ and $C_1$ do not depend on $h$.*

*Proof.* Taking into account that

$$\psi_{k+e_j} - \psi_k = h\partial_j^+\psi_k\,, \tag{5.14}$$

we obtain

$$|\psi_{k+e_j}|^2\overline{\psi}_{k+e_j} - |\psi_k|^2\overline{\psi}_k = |\psi_{k+e_j}|^2 \left(\overline{\psi}_{k+e_j} - \overline{\psi}_k\right) + \overline{\psi}_k \left(|\psi_{k+e_j}|^2 - |\psi_k|^2\right)$$

$$= |\psi_{k+e_j}|^2 h\overline{\partial_j^+\psi_k} + \overline{\psi}_k \left(\overline{\psi}_{k+e_j}(\psi_{k+e_j} - \psi_k) + \psi_k(\overline{\psi}_{k+e_j} - \overline{\psi}_k)\right)$$

and therefore

$$\begin{aligned}
&\mathrm{Re}\left\{\left(\partial_j^+\psi_k\right)\partial_j^+(|\psi_k|^2\overline{\psi}_k)\right\}\\
&\quad= |\psi_{k+e_j}|^2|\partial_j^+\psi_k|^2 + \mathrm{Re}\left((\partial_j^+\psi_k)^2\overline{\psi}_k\overline{\psi}_{k+e_j}\right) + |\psi_k|^2|\partial_j^+\psi_k|^2\\
&\quad\geqslant |\partial_j^+\psi_k|^2 \left(|\psi_{k+e_j}|^2 + |\psi_k|^2 - |\psi_k||\psi_{k+e_j}|\right)\\
&\quad\geqslant \frac{3}{4}|\partial_j\psi_k|^2|\psi_k|^2\,.
\end{aligned} \tag{5.15}$$

In addition,

$$\begin{aligned}
\mathrm{Im}\sum_j \left(A_k^j(\partial_j^+\psi_k)|\psi_k|^2\overline{\psi}_k\right) &\geqslant -|A_k||\nabla^+\psi_k||\psi_k|^3\\
&\geqslant -C_\varepsilon|\psi_k|^4 - \varepsilon|\nabla^+\psi_k|^2|\psi_k|^2
\end{aligned} \tag{5.16}$$

so that

$$\begin{aligned}
&\mathrm{Im}\left(\psi_k(A_k, \nabla_h^+)(|\psi_k|^2\overline{\psi}_k)\right)\\
&\quad= \mathrm{Im}\left(\psi_k\sum_j \frac{A_k^j}{h}(|\psi_{k+e_j}|^2\overline{\psi}_{k+e_j} - |\psi_k|^2\overline{\psi}_k)\right)\\
&\quad= \mathrm{Im}\left(\psi_k\sum_j \frac{A_k^j}{h}\left((\psi_{k+e_j} - \psi_k)\overline{\psi}_{k+e_j}^2\right.\right.\\
&\qquad\qquad \left.\left. + \psi_k(\overline{\psi}_{k+e_j} - \overline{\psi}_k)(\overline{\psi}_{k+e_j} + \overline{\psi}_k)\right)\right)
\end{aligned} \tag{5.17}$$

$$= \operatorname{Im}\Big(\sum_j A_k^j\big((\partial_j^+\psi_k)\psi_k\overline{\psi}_{k+e_j}^2 + \psi_k^2\overline{\partial_j^+\psi_k}(\overline{\psi}_{k+e_j}+\overline{\psi}_k)\big)\Big)$$

$$\geqslant -C\sum_j|\partial_j^+\psi_k||\psi_k|\Big(|\psi_{k+e_j}|^2+|\psi_k||\psi_{k+e_j}|+|\psi_k|^2\Big)$$

$$\geqslant -\varepsilon\sum_j|\partial_j^+\psi_k|^2|\psi_k|^2 - C_\varepsilon\sum_j\Big(|\psi_{k+e_j}|^4+|\psi_k|^4\Big).$$

Using (5.15)–(5.17), we obtain

$$\begin{aligned}&\operatorname{Re}\Big((i\nabla_h^+ + A_k)\psi_k, \overline{(i\nabla_h^+ + A_k)(|\psi_k|^2\psi_k)}\Big)\\ &= \operatorname{Re}(\nabla_h^+\psi_k, \nabla_h^+(|\psi_k|^2\overline{\psi}_k)) - \operatorname{Im}(\nabla_h^+\psi_k, A_k|\psi_k|^2\overline{\psi}_k)\\ &\quad+\operatorname{Im}(\psi_k(A_k\nabla_h^+)(|\psi_k|^2\overline{\psi}_k)) + |A_k|^2|\psi_k|^4\\ &\geqslant \frac{3}{4}\sum_{j=1}^d|\partial_j^+\psi_k|^2|\psi_k|^2 - C_\varepsilon(|\psi_k|^4+\sum_{j=1}^d|\psi_{k+e_j}|^4) - \varepsilon|\nabla_h^+\psi_k|^2|\psi_k|^2\,.\end{aligned} \tag{5.18}$$

Now we substitute (5.18) into (5.3) and subsequently use this inequality in (5.2). As a result, taking into account (5.8), we obtain the inequality

$$\begin{aligned}d\|\boldsymbol{\psi}\|_{L^{4,h}}^4 \leqslant h^d\widetilde{\sum_{jk}}&\Big(-(3-4\varepsilon)|\partial_{j,h}^+\psi_k|^2|\psi_k|^2\Big)dt\\ &+\sum_k\left\{C|\psi_k|^4+\varepsilon|\psi_k|^2-4|\psi_k|^6\right\}dt\\ &+\sum_k\Big(2\operatorname{Re}(\overline{\psi}_k\widehat{r}[\psi_k]dW_k)+C\big(|\psi_k|^2+|\psi_k|^4\big)dt\Big)\,.\end{aligned} \tag{5.19}$$

Rewriting (5.19) in integral form and taking the mathematical expectation of the obtained inequality, we obtain the estimate

$$\begin{aligned}E\|\boldsymbol{\psi}(t)\|_{L^{4,h}}^4 + E\int_0^t h^d\left(\widetilde{\sum_{jk}}|\partial_{j,h}^+\psi_k|^2|\psi_k|^2+\sum_k|\psi_k|^6\right)d\tau\\ \leqslant CE\int_0^t h^d\sum_k\left(|\psi_k|^4+|\psi_k|^2+1\right)d\tau + E\|\boldsymbol{\psi}_0\|_{L^4}^4\,.\end{aligned} \tag{5.20}$$

Applying the bound (5.6) to the right-hand side of (5.20) and applying after that the Gronwall inequality, we obtain the final estimate (5.13). □

Note that, in addition to the estimates (5.6) and (5.13) corresponding to the cases $p = 1$ and $p = 2$, one can prove by induction analogous estimates for arbitrary natural $p$; specifically, we have

$$E\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} + \int_0^t \left( \|\boldsymbol{\psi}(\tau)\|^{2(p+1)}_{L^{2(p+1),h}} + h^d \widetilde{\sum_{jk}} |\partial^+_{j,h}\psi_k|^2 |\psi_k|^{2(p-1)} \right) d\tau \leqslant C_p \left(1 + E\|\boldsymbol{\psi}_0(t)\|^p_{L^{p,h}}\right) e^{Ct} . \tag{5.21}$$

We will not prove the estimate (5.21) for $p \geqslant 3$ because, for our purposes, the estimates (5.6) and (5.20) will suffice.

## *5.4 Auxiliary Wiener process*

We will need a more general projection of the initial Wiener process than (4.11). Roughly speaking, the new projection contains not only coordinates from (4.11), but also their difference gradients at points $kh$. To be precise, in a manner similar to (4.9), we define for, $f(x) \in L^2(G)$,

$$p^0_k(f) = V^{-1}(Q_k) \int_{Q_k} f(x)\, dx, \quad \text{for } kh \in G_h\,, \tag{5.22}$$

$$p^0_k(f) \text{ for } kh \in \partial G^+_h \text{ is calculated by } p^0_k(f) \text{ with } kh \in \partial G^-_h \text{ using (2.23)}\,, \tag{5.23}$$

and

$$p^j_k(f) = i\frac{p^0_{k+e_j}(f) - p^0_k(f)}{h} + A^j_k p^0_k(f) \quad \text{for } kh \in G_h \cup \partial G^+_h(-j) \tag{5.24}$$

for $j = 1, \ldots, d$. We denote

$$\widehat{G_h} = \{(j,k) \,:\, j = 0, kh \in G_h;\, j = 1, \ldots, d, kh \in G_h \cup \partial G^+_h(-j)\}$$

and introduce the projector

$$P^A_h : L^2(G) \to L^2(\widehat{G_h}); \quad P^A_h(f) = \{p^j_k(f), (j,k) \in \widehat{G_h}\}\,, \tag{5.25}$$

where the scalar product in $L^2(\widehat{G_h})$ is defined in the standard way:

$$\text{for} \quad u = \{u^j_k, (j,k) \in \widehat{G_h}\}, \quad v = \{v^j_k, (j,k) \in \widehat{G_h}\},$$

$$(u,v)_{L^2(\widehat{G}_h)} = h^d \sum_{(j,k)\in\widehat{G}_h} u_k^j \overline{v_k^j}\,.$$

Note that the components $u_k^j$ of $u \in L^2(\widehat{G}_h)$ with $j \neq 0$ are expressed via the components $u_m^0$ by the formula analogous to (5.24):

$$u_k^j = \frac{u_{k+e_j}^0 - u_k^0}{h} + A_k^j u_k^0 \quad \text{for } j = 1,\dots,d, \quad kh \in G_h \cup \partial G_h^+(-j)\,.$$

We can calculate the operator $(P_h^A)^* : L^2(\widehat{G}_h) \to L^2(G)$ which is adjoint to (5.25) using (2.14) which is summation by parts:

$$\begin{aligned}(P_h^A f, g)_{L^2(\widehat{G}_h)} &= h^d \sum_{(j,k)\in\widehat{G}_h} p_k^j(f) g_k^j \\ &= h^d \widetilde{\sum_{jk}} (i\partial_{j,h}^+ + A_k^j) p^0(f) \overline{(i\partial_{j,h}^+ + A_h^j) g_k^0} + h^d \sum_{kh\in G_h} p_k^0(f) \overline{g_k^0} \\ &= h^d \sum_{kh\in G_h} p_k^0(f) \Big( \overline{g_k^0 + (i\nabla_h^- + A_k, i\nabla_h^+ + A_k) g_k^0} \Big)\end{aligned} \tag{5.26}$$

so that

$$\big((P_h^A)^* g\big)(x) = \sum_{kh\in G_h} h^d V^{-1}(Q_k) \big(g_k^0 + (i\nabla_h^- + A_k, i\nabla_h^+ + A_k) g_k^0\big) \mathcal{X}_k(x)\,. \tag{5.27}$$

Analogous to (4.11), we introduce the vector-valued process

$$\mathbf{AW}(t,\omega) = P_h^A W(t,\cdot,\omega) = \{p_k^j(W(t,\cdot,\omega) \equiv AW_k^j(t), (j,k) \in \widehat{G}_h\}\,. \tag{5.28}$$

Here, $p_k^0(W(t,\cdot,\omega)) = W^k(t,\omega)$ for $kh \in G_h$, where $W^k(t,\omega)$ is the Wiener process from (4.11). In order to define $p_k^j(W(t,\cdot,\omega))$ by (5.24), one has to know $W^k(t,\omega)$ with $kh \in \partial G^+$. These Wiener processes are defined by formula (2.23) via $W^m(t,\omega)$ with $mh \in G_h$. Repeating the calculation in (4.13) and (4.15), where the projector (4.9) is changed to the projector (5.25) and $L^{2h}$ is changed to $L^2(\widehat{G}_h)$, we find that the process (5.28) is a vector-valued Wiener process. Moreover,

$$\begin{aligned}&B\big((P_h^A)^* v, (P_h^A)^* v\big) \\ &= \int_0^\infty \int_0^\infty t \wedge s \int_{G\times G} \mathcal{K}(x,y) (P_h^A)^*(v(s))(y) \overline{(P_h^A)^*(v(t))(x)}\, dxdydsdt\end{aligned}$$

$$= \int_0^\infty \int_0^\infty t \wedge s \sum_{(j_1,k_1)\in\widehat{G}_h} \sum_{(j_2,k_2)\in\widehat{G}_h} \mathcal{K}^{j_1,j_2}_{k_1,k_2} \overline{v^{j_1}_{k_1}(t)} v^{j_2}_{k_2}(s)\, dtds\,, \tag{5.29}$$

where $\mathcal{K}^{0,0}_{j,k}$ are defined by (4.17) with the upper indices (0,0) omitted,

$$\mathcal{K}^{j_1,j_2}_{k_1,k_2} = \big(i\partial^+_{j_1,h} + A^{j_1}_{k_1}\big)\overline{\big(i\partial^+_{j_2,h} + A^{j_2}_{k_2}\big)}\mathcal{K}^{0,0}_{k_1,k_2}, \quad (j_\ell, k_\ell) \in \widehat{G}_h, j_\ell \neq 0, \ell = 1,2\,, \tag{5.30}$$

and $\mathcal{K}^{j_1,0}_{k_1,k_2}$ and $\mathcal{K}^{0,j_2}_{k_1,k_2}$ are defined similarly (correspondingly, the second or first operator $\big(i\partial^+_{j,h} + A^j_k\big)$ in (5.30) should be omitted). It is clear that

$$s \wedge t\, \mathcal{K}^{j_1,j_2}_{k_1,k_2} = \int AW^{j_1}_{k_1}(t,\omega)\overline{AW^{j_2}_{k_2}(s,\omega)}m(d\omega)\,, \tag{5.31}$$

where the scalar Wiener processes $AW^j_k(t,\omega)$ are defined in (5.28). Definitions (4.17), (5.30), and (5.31) of the operator $A\mathcal{K} = \{h^d \mathcal{K}^{j_1,j_2}_{k_1,k_2}\}$ imply that $\mathcal{K}^{j_1,j_2}_{k_1,k_2} = \overline{\mathcal{K}^{j_2,j_1}_{k_2,k_1}}$ and therefore there exists a unitary transformation $A\theta = \{\theta^{j_1,j_2}_{k_1,k_2}\}$ (i.e., $A\theta^* \equiv \{\overline{\theta^{j_2,j_1}_{k_2,k_1}}\} = (A\theta)^{-1}$) that reduces the operator $A\mathcal{K}$ to diagonal form:

$$A\theta^* A\mathcal{K} A\theta = AL, \quad \text{where} \quad AL = \{L^{j_1,j_2}_{k_1,k_2}\} = \{\delta_{j_1,j_2}\delta_{k_1,k_2}\mu^{j_1}_{k_1}\}\,. \tag{5.32}$$

We set

$$\widetilde{\mathbf{AW}}(t,\omega) = A\theta^* \mathbf{AW}(t,\omega) = \{\widetilde{W}^j_k(t,\omega)\}\,. \tag{5.33}$$

Then calculations analogous to (4.25) and (4.26) show that the scalar Wiener processes $\widetilde{W}^j_k(t,\omega)$ are independent and therefore, for their differentials, the following Ito table analogous to (4.36) is true:

$$\begin{cases} d\mathrm{Re}\,\widetilde{W}^{j_1}_{k_1}(t) d\mathrm{Re}\,\widetilde{W}^{j_2}_{k_2}(t) = d\mathrm{Im}\,\widetilde{W}^{j_1}_{k_1}(t) d\mathrm{Im}\,\widetilde{W}^{j_2}_{k_2}(t) = \mu^{j_1}_{k_1}\delta_{j_1,j_2}\delta_{k_1,k_2} dt \\ \\ d\mathrm{Re}\,\widetilde{W}^j_k(t)dt = d\mathrm{Im}\,W^j_k(t)dt = d\mathrm{Re}\,W^{j_1}_{k_1}(t) d\mathrm{Im}\,W^{j_2}_{k_2}(t) = 0\,. \end{cases} \tag{5.34}$$

Now we are in a position to prove the following analogue of (4.39).

**Lemma 5.3.** *For scalar Wiener processes $AW^j_k(t)$ defined in* (5.28) *the following relationships hold*:

$$(d\mathrm{Re}\,AW^j_k)^2 = (d\mathrm{Im}\,AW^j_k)^2 = \sum_{(m,\ell)\in\widehat{G}_h} \mu^\ell_m |\theta^{j\ell}_{km}|^2 dt, \quad d\mathrm{Re}\,AW^j_k\; d\mathrm{Im}\,AW^j_k = 0\,. \tag{5.35}$$

*Proof.* By virtue of (5.33),

$$AW_k^j = \sum_{(\ell,m)\in\widehat{G_h}} \theta_{k,m}^{j,\ell}\widetilde{W}_m^\ell\,. \tag{5.36}$$

Therefore, taking into account (5.34) and the fact that the transformation $A\theta = \|\theta_{k,m}^{j,l}\|$ is unitary, we obtain

$$\begin{aligned}
\left(d\mathrm{Re}\,AW_k^j\right)^2 &= \Big(\sum_{\ell,m}\Big[\mathrm{Re}\,\theta_{k,m}^{j,\ell}d\mathrm{Re}\,\widetilde{W}_m^\ell - \mathrm{Im}\,\theta_{k,m}^{j,\ell}d\mathrm{Im}\,\widetilde{W}_m^\ell\Big]\Big)^2\\
&= \sum_{\ell,m}\sum_{\ell_1,m_1}\Big(\mathrm{Re}\,\theta_{k,m}^{j,\ell}\mathrm{Re}\,\theta_{k,m_1}^{j,\ell_1}d\mathrm{Re}\,\widetilde{W}_m^\ell d\mathrm{Re}\,\widetilde{W}_{m_1}^{\ell_1}\\
&\qquad +\mathrm{Im}\,\theta_{k,m}^{j,\ell}\mathrm{Im}\,\theta_{k,m_1}^{j,\ell_1}d\mathrm{Im}\,\widetilde{W}_m^\ell d\mathrm{Im}\,\widetilde{W}_{m_1}^{\ell_1}\Big)\\
&= \sum_{\ell,m}\mu_m^\ell|\theta_{k,m}^{j,\ell}|^2\,dt\,.
\end{aligned} \tag{5.37}$$

The other relations in (5.35) are proved in a similar manner. □

**Lemma 5.4.** *The following equalities hold*:

$$\begin{cases}
d\mathrm{Re}\,AW_k^j d\mathrm{Re}\,AW_k^0 = d\mathrm{Im}\,AW_k^j d\mathrm{Im}\,AW_k^0 = \sum_{l,m}\mu_m^\ell\mathrm{Re}\left(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\right)\,dt\\
d\mathrm{Re}\,AW_k^j d\mathrm{Im}\,AW_k^0 = 0\,,
\end{cases} \tag{5.38}$$

*where* $j = 1,\ldots,d$.

*Proof.* Similar to (5.37), we find

$$\begin{aligned}
d\mathrm{Re}\,AW_k^j d\mathrm{Re}\,AW_k^0 &= \sum_{\ell,m}\mu_m^\ell\Big(\mathrm{Re}\,\theta_{k,m}^{j,\ell}\mathrm{Re}\,\theta_{k,m}^{0,\ell} + \mathrm{Im}\,\theta_{k,m}^{j,\ell}\mathrm{Im}\,\theta_{k,m}^{0,\ell}\Big)\,dt\\
&= \sum_{\ell,m}\mu_m^\ell\mathrm{Re}\left(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\right)\,dt\,.
\end{aligned} \tag{5.39}$$

All the other equalities from (5.38) are proved in a similar manner. □

We will need the following lemma.

**Lemma 5.5.** *Let* $\mu_m^\ell$ *denote the eigenvalues from* (5.32). *Then* $\mu_m^\ell \geqslant 0$ *and the following estimates hold*:

$$\sum_{(\ell,m)\in\widehat{G_h}} \mu_m^\ell|\theta_{k,m}^{j,\ell}|^2 \leqslant \sum_{(\ell,m)\in\widehat{G_h}} \mu_m^\ell\,, \tag{5.40}$$

$$\sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell \operatorname{Re}\big(\theta_{k,m}^{j,\ell}\overline{\theta_{j,m}^{0,\ell}}\big) \leqslant \sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell\,. \tag{5.41}$$

*Proof.* To show that $\mu_m^\ell \geqslant 0$, we have to prove that the operator $A\mathcal{K} = \{h^d\mathcal{K}_{k_1,k_2}^{j_1,j_2}\}$ is positive semi-definite. Let $v = \{v_k^j\} \in L^2(\widehat{G}_h)$. Then, by virtue of (5.30) and (2.14), we obtain

$$\begin{aligned}
&h^{2d}\sum_{(j,k)\in\widehat{G}_h}\sum_{(\ell,m)\in\widehat{G}_h}\mathcal{K}_{k,m}^{j,\ell}v_m^\ell\overline{v_k^j} = h^{2d}\sum_{\substack{kh\in G_h\\ mh\in G_h}}\mathcal{K}_{k,m}^{0,0}v_m^0\overline{v_k^0}\\
&\quad+h^{2d}\widetilde{\sum_{j,k}}\widetilde{\sum_{\ell,m}}\Big((i\partial_{j,h}^+ + A_k^j)(\overline{i\partial_{\ell,h}^+ + A_m^\ell})\mathcal{K}_{k,m}^{0,0}\Big)(i\partial_{\ell,h}^+ + A_m^\ell)v_m^0\overline{(i\partial_{j,h}^+ + A_k^j)v_k^0}\\
&\quad+h^{2d}\widetilde{\sum_{j,k}}\sum_{mh\in G_h}\Big((i\partial_{jh}^+ + A_k^j)\mathcal{K}_{k,m}^{0,0}v_m^0\overline{(i\partial_{j,h}^+ + A_k^j)v_k^0}\\
&\quad+h^{2d}\sum_{kh\in G_h}\widetilde{\sum_{\ell,m}}(\overline{i\partial_{\ell,h}^+ + A_m^\ell})\mathcal{K}_{k,m}^{0,0}(i\partial_{\ell,h}^+ + A_m^\ell)v_m^0\overline{v_k^0}\\
&= h^{2d}\sum_{kh\in G_h}\sum_{mh\in G_h}\mathcal{K}_{k,m}^{0,0}\big(1+(i\nabla_m + A_m)^2\big)v_m^0\overline{\big(1+(i\nabla_k + A_k)^2\big)v_k^0}\\
&= \int\limits_{G\times G}\mathcal{K}(x,y)\big((P_h^A)^*v\big)(y)\overline{\big(P_h^A)^*v\big)(x)}\,dxdy \geqslant 0
\end{aligned} \tag{5.42}$$

because the positive semi-definiteness of the operator $\mathcal{K}$ was assumed in (3.5). To prove (5.40), it is enough to note that since the matrix $\{\theta_{k,m}^{j,\ell}\}$ is unitary, we have

$$\sum_{(\ell,m)\in\widehat{G}_h}\big|\theta_{k,m}^{j,\ell}\big|^2 = \sum_{(\ell,m)\in\widehat{G}_h}\theta_{k,m}^{j,\ell}\big(\theta_{m,k}^{\ell,j}\big)^* = 1$$

and therefore $|\theta_{k,m}^{j,\ell}|^2 \leqslant 1$ for each $(j,k)$, $(\ell,m)$. Thus

$$\big|\operatorname{Re}\big(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\big)\big| \leqslant \big|\theta_{k,m}^{j,\ell}\big|\,\big|\theta_{k,m}^{0,\ell}\big| \leqslant 1$$

which implies (5.40) and (5.41). □

**Lemma 5.6.** *The following bound is valid:*

$$\sum_{(\ell,m)\in\widehat{G}_h}\mu_m^\ell \leqslant C\Big(\int\limits_G \mathcal{K}(x,x)\,dx + \sum_{j=1}^d\int\limits_G \partial_{x_j}\partial_{y_j}\mathcal{K}(x,y)\Big|_{y=x}dx + 1\Big), \tag{5.43}$$

*where the constant $C$ does not depend on $h$ and $\mathcal{K}(x,x)$ is the kernel* (3.14).

*Proof.* By virtue of (4.17) and (5.30), we have

$$\sum_{(j,k)\in\widehat{G_h}} \mu_k^j = h^d \sum_{(j,k)\in\widehat{G_h}} \mathcal{K}_{k,k}^{j,j} = I_1 + I_2\,, \tag{5.44}$$

where

$$I_1 = h^d \sum_k \mathcal{K}_{k,k}^{0,0}, \qquad I_2 = h^d \widetilde{\sum_{jk}} \mathcal{K}_{k,k}^{j,j}\,. \tag{5.45}$$

By virtue of Lemma 4.1, we have

$$I_1 = \sum_{jh\in G_h} \mu_j \leqslant C \int\limits_G \mathcal{K}(x,x)\, dx\,. \tag{5.46}$$

From (5.30), we have

$$I_2 = h^d \widetilde{\sum_{(j,k)}} \Big( \big(i\partial_{j,h}^+ + A_k^j\big) \overline{\big(i\partial_{j,h}^+ + A_m^j\big)} \mathcal{K}_{k,m}^{0,0} \Big)\Big|_{m=k}\,. \tag{5.47}$$

Note that, in fact, the summation in (5.47) is performed over $(j,k)$ such that $kh \in G_h$ and $(k+e_j)h \in G_h$ because, by virtue of (5.23) and (2.15), all other summands in (5.47) vanish. Therefore, taking into account that $\mathcal{K}_{k,m}^{0,0} = \mathcal{K}_{km}$ is defined by (4.17) and after changing variables in the integrals (4.17) in the appropriate terms connected with $i\partial_{j,h}^+ \mathcal{K}_{k,m}^{0,0}$, we obtain

$$I_2 = \int\limits_{G^0(h)} \big(i\partial_{j,h}^-(x) + A^j(x)\big) \overline{\big(i\partial_{j,h}^-(y) + A^j(y)\big)} \mathcal{K}(x,y)\big|_{y=x} dx + J\,. \tag{5.48}$$

Here, $\partial_{j,h}^-(x)\mathcal{K}(x,y) = \big(\mathcal{K}(x,y) - \mathcal{K}(x-e_jh, y)\big)/h$, $\partial_{j,h}^-(y)\mathcal{K}(x,y) = \big(\mathcal{K}(x,y) - \mathcal{K}(x, y-e_jh)\big)/h$, and $G^0(h) = \sum_{kh\in G_h^0} Q_k$, where $G_h^0 = G_h \backslash \partial^- G_h$ (see Definition 2.1 in Sect. 2.2), and $Q_k$ are the sets are defined in (4.1). The term $J$ arises because of the summation of some terms connected with $\mathcal{K}_{k,k}^0$ with $kh \in \partial^- G_h$. It is easy to see that

$$|J| \leqslant C\,, \tag{5.49}$$

where $C$ does not depend on $h$. Using the representation

$$\mathcal{K}(x,y) = \sum_r \lambda_r e_r(x) e_r(y),$$

we obtain from (5.48) and (5.49)

$$|I_2| \leqslant \sum_r \lambda_r \int\limits_{G^0(h)} |\big(i\nabla_h^- + A(x)\big)e_r(x)|^2\, dx + |J|$$

$$\leqslant C\Big(1 + \sum_r \lambda_r \int\limits_{G^0(h)} \big(|\nabla_h^- e(x)|^2 + |e(x)|^2\big)\, dx\Big) \qquad (5.50)$$

$$\leqslant C\Big(1 + \sum_r \lambda_r \int\limits_G \big(|\nabla e(x)|^2 + |e(x)|^2\big)\, dx\Big)\,,$$

where the last inequality estimating the finite difference by the derivative can be obtained by using the elementary equality $u(x+h) - u(x) = \int\limits_x^{x+h} u'(y)\, dy$. The bounds (5.46) and (5.50) imply (5.43). □

## 5.5 A priori estimates for $\Delta_h \psi_k$

In addition to (5.1) and (5.5), we introduce the notation

$$\|\Delta_h \psi\|^2_{L^{2,h}} = \sum_{kh \in G_h} |\Delta_h \psi_k|^2\,, \qquad (5.51)$$

where the values $\psi_k$ with $kh \in \partial G_h^+$ (we need these values to define $\Delta_h \psi_k$) are defined with the help of (2.23). We will also need the following estimate.

**Theorem 5.7.** *Let a random process* $\boldsymbol{\psi}(t) = \{\psi_k\}$ *have the stochastic differential* (4.30). *Then* $\boldsymbol{\psi}$ *satisfies the bound*

$$E\Big(\|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \int\limits_0^t \|\Delta_h \boldsymbol{\psi}(\tau)\|^2_{L^{2,h}}\, d\tau\Big) \qquad (5.52)$$

$$\leqslant E(\|\nabla_h^+ \boldsymbol{\psi}_0\|^2_{L^{2,h}}) + C_3 e^{Ct}\Big(E(\|\boldsymbol{\psi}_0\|^4_{L^{4,h}}) + 1\Big)\,,$$

*with constants* $C_3$ *and* $C$ *independent of* $h$.

*Proof.* We apply the Ito formula to the function $u(\boldsymbol{\psi}) = h^d \widetilde{\sum_{jk}} |(i\partial_{j,h}^+ + A_k^j)\psi|^2$ to obtain

$$du(\boldsymbol{\psi}) = h^d \widetilde{\sum_{j,k}} \big((i\partial_{j,h}^+ + A_k^j) d\psi_k, \overline{(i\partial_{j,h}^+ + A_k^j)\psi_k}\big)$$

$$+h^d\widetilde{\sum_{j,k}}\big((i\partial^+_{j,h}+A^j_k)\psi_k,\overline{(i\partial^+_{j,h}+A^j_k)d\psi_k}\big)$$

$$+\frac{h^d}{2}\widetilde{\sum_{j,k}}\big((i\partial^+_{j,h}+A^j_k)d\psi_k,\overline{(i\partial^+_{j,h}+A^j_k)d\psi_k}\big) \tag{5.53}$$

$$=\mathrm{I}+\bar{\mathrm{I}}+\mathrm{II}\,,$$

where I, $\bar{\mathrm{I}}$, and II are the first, second and third terms of the right-hand side of (5.53) respectively. Applying (2.14) and (4.30) and using the notation $(i\nabla_h^-+A_k,i\nabla_h^++A_k)=(i\nabla_h+A_k)^2$, we obtain

$$\mathrm{I}=h^d\sum_k d\psi_k\overline{(i\nabla+A_k)^2\psi_k}=\Big\{-h^d\sum_k|(i\nabla+A_k)^2\psi_k|^2$$
$$+h^d\widetilde{\sum_{j,k}}|(i\partial^+_{j,h}+A^j_k)\psi_k|^2-\big((i\partial^+_{j,h}+A^j_k)(|\psi_k|^2\psi_k),\overline{(i\partial^+_{j,h}+A^j_k)\psi_k}\big)\Big\}dt$$
$$+h^d\sum_k\Big\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\Big\}. \tag{5.54}$$

Since $\bar{\mathrm{I}}$ is the complex conjugate to I, we obtain from (5.54) that

$$\mathrm{I}+\bar{\mathrm{I}}=-2h^d\sum_{kh\in G_h}|(i\nabla+A_k)^2\psi_k|^2dt+\widetilde{\sum_{j,k}}C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)dt$$
$$+h^d\sum_{kh\in G_h}\Big\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\Big\}\,, \tag{5.55}$$

where $C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)$ admits the bound

$$|C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)|\leqslant C\Big(|\psi_k|^6+|\partial^+_{j,h}\psi_k|^2|\psi_k|^2+|\partial^+_{j,h}\psi_k|^2+1\Big) \tag{5.56}$$

with constant $C$ independent of $j,k,h$.

Let us consider the term II. Applying (2.14), (4.30), and using the notation $\mathcal{D}_kdt$ for the term with the differential $dt$ in (4.30) and taking into account (4.36) and (4.25), we have

$$2\mathrm{II}=h^d\widetilde{\sum_{jk}}\Big((i\partial^+_{jh}+A^j_k)(\mathcal{D}_kdt+\widehat{r}[\psi_k]dW_k)$$

$$\overline{(i\partial_{jh}^{+} + A_k^j)(\mathcal{D}_k dt + \widehat{r}[\psi_k]dW_k)}\Big)$$

$$= h^d \widetilde{\sum_{jk}} \Big( \big(i\partial_{jh}^{+} + A_k^j\big)\big(\widehat{r}[\psi_k]dW_k\big)\overline{\big(i\partial_{jh}^{+} + A_k^j\big)\big(\widehat{r}[\psi_k]dW_k\big)}\Big) \,. \tag{5.57}$$

Using the equality

$$a_{k+1}b_{k+1} - a_k b_k = a_{k+1}(b_{k+1} - b_k) + (a_{k+1} - a_k)b_k$$

and the definitions (3.20) and (3.21), we obtain

$$(i\partial_{j,h}^{+} + A_k^j)(\widehat{r}[\psi_k]dW_k)$$

$$= \widehat{r}[i\overline{\psi}_{k+e_j}](i\partial_{j,h}^{+} + A_k^i)dW_k + (i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k])dW_k \,. \tag{5.58}$$

Using (5.58) and the definition (5.28) of the scalar Wiener process $AW_k^j(t)$, we obtain, from (5.57),

$$2II = h^d \widetilde{\sum_{j,k}} \Big| \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j + \big(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k]\big)dAW_k^0 \Big|^2$$

$$= J_1 + J_2 + J_3 \,, \tag{5.59}$$

where

$$J_1 = h^d \widetilde{\sum_{j,k}} \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j \; \overline{\widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j} \,, \tag{5.60}$$

$$J_2 = h^d \widetilde{\sum_{j,k}} 2\mathrm{Re}\,\Big\{ \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j \overline{(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k])dAW_k^0} \Big\} \,, \tag{5.61}$$

and

$$J_3 = h^d \widetilde{\sum_{j,k}} \big| \big(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k]\big)dAW_k^0 \big|^2 \,. \tag{5.62}$$

By virtue of (3.20), (3.21), and (5.35), we obtain from (5.60) that

$$J_1 = h^d \widetilde{\sum_{j,k}} \big( r^2(\mathrm{Im}\,\psi_{k+e_j}) + r^2(\mathrm{Re}\,\psi_{k+e_j}) \big) \sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell |\theta_{k,m}^{j,\ell}|^2 dt \,. \tag{5.63}$$

Similarly, (5.38) and (5.61) imply

$$J_2 = 2h^d \widetilde{\sum_{j,k}} \Big\{ r(\operatorname{Im}\psi_{k+e_j}) d\operatorname{Re} AW_k^j \big( -(\partial_{j,h}^+ r(\operatorname{Im}\psi_k)) d\operatorname{Im} AW_k^0$$

$$+ A_k^j (r(\operatorname{Re}\psi_k) - r(\operatorname{Im}\psi_{k+e_j})) d\operatorname{Re} AW_k^0 \big) + r(\operatorname{Re}\psi_{k+e_j}) d\operatorname{Im} AW_k^j$$

$$\cdot \Big( -\partial_{j,h}^+ r(\operatorname{Re}\psi_k) d\operatorname{Re} AW_k^0 + A_k^j (r(\operatorname{Im}\psi_k) - r(\operatorname{Re}\psi_{k+e_j})) d\operatorname{Im} AW_k^0 \Big) \Big\}$$

$$= 2h^d \widetilde{\sum_{j,k}} \Big\{ A_k^j \Big( r(\operatorname{Im}\psi_{k+e_j}) r(\operatorname{Re}\psi_k) - r^2(\operatorname{Im}\psi_{k+e_j})$$

$$+ r(\operatorname{Re}\psi_{k+e_j}) r(\operatorname{Im}\psi_k) - r^2(\operatorname{Re}\psi_{k+e_j}) \Big) \sum_{(\ell,m)\in\widehat{G_h}} \mu_m^\ell \operatorname{Re}(\theta_{k,m}^{j,\ell} \overline{\theta_{k,m}^{0,\ell}}) \Big\} dt\,. \tag{5.64}$$

In addition, by (5.35) and (5.62), we have

$$J_3 = h^d \widetilde{\sum_{j,k}} \Big\{ \big( -\partial_{j,h}^+ r(\operatorname{Im}\psi_k) d\operatorname{Im} AW_k^0$$

$$+ A_k^j \big( r(\operatorname{Re}\psi_k) - r(\operatorname{Im}\psi_{k+e_j}) \big) d\operatorname{Re} AW_k^0 \big)^2$$

$$+ \big( \partial_{j,h}^+ r(\operatorname{Re}\psi_k) d\operatorname{Re} AW_k^0 + A_k^i (r(\operatorname{Im}\psi_k) - r(\operatorname{Re}\psi_{k+e_j})) d\operatorname{Im} AW_k^0 \big)^2 \Big\}$$

$$= h^d \widetilde{\sum_{j,k}} \Big\{ \big( \partial_{j,h}^+ r(\operatorname{Im}\psi_k) \big)^2 + \big( \partial_{j,h}^+ r(\operatorname{Re}\psi_k) \big)^2$$

$$+ (A_k^j)^2 \big( (r(\operatorname{Re}\psi_k) - r(\operatorname{Im}\psi_{k+e_j}))^2$$

$$+ (r(\operatorname{Im}\psi_k) - r(\operatorname{Re}\psi_{k+e_j}))^2 \Big\} \sum_{\ell,m} \mu_m^\ell |\theta_{k,m}^{0,\ell}|^2 dt\,. \tag{5.65}$$

Now relations (5.59), (5.63), (5.64), and (5.65) and Lemmas 5.5 and 5.6 imply that

$$II = \widetilde{\sum_{j,k}} d_{jk}(\psi_k, \psi_{k+e_j} . \partial_{j,h}^+ \psi_k) dt\,, \tag{5.66}$$

where

$$|d_{jk}(\psi_k, \psi_{k+e_j}, \partial_{j,h}^+ \psi_k)| \leqslant C\big(1 + |\psi_k|^2 + |\psi_{k+e_j}|^2 + |\partial_{j,h}^+ \psi_k|^2\big) \tag{5.67}$$

with constant $C$ independent of $j, k, h$.

Relations (5.53), (5.55), and (5.66) give

$$d\Big(h^d\widetilde{\sum_{j,k}}|(i\partial^+_{j,h}+A^j_k)\psi_k|^2\Big)+2h^d\sum_{kh\in G_h}|(i\nabla^+_h+A_k)^2\psi_k|^2$$
$$=\widetilde{\sum_{j,k}}\big(C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)+d_{jk}(\psi_k,\psi_{k+e_j},\partial^+_{j,h}\psi_k)\big)dt \tag{5.68}$$
$$+h^d\sum_{khG_h}\big\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\big\}\,.$$

Writing the differential Ito formula (5.68) in integral form and applying the mathematical expectation, we obtain

$$E\Big(\|(i\nabla^+_h+\mathbf{A})\boldsymbol{\psi}(t)\|^2_{L^{2,h}}+2\int_0^t\|(i\nabla_h+A)^2\boldsymbol{\psi}(\tau)\|^2_{L^{2,h}}\,d\tau\Big)$$
$$=E\Big(\int_0^t h^d\widetilde{\sum_{j,k}}\Big\{(C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)+d_{jk}(\psi_k,\psi_{k+e_j},\partial^+_{j,h}\psi_k)\Big\}\,d\tau\Big) \tag{5.69}$$
$$+E\big(\|(i\nabla^+_h+A)\boldsymbol{\psi}_0\|^2_{L^{2,h}}\big)\,,$$

where $\mathbf{A}=\{A^j_k,kh\in G_h\cup\partial G^+_h(-j)\}$. Doing a simple transformation with the first term on the left-hand side of (5.69), applying the bounds (5.56) and (5.67) to the right-hand side of (5.69), and then applying to the result the inequalities (5.6) and (5.13) results in (5.52). □

# 6 Existence Theorem for Approximations

The aim of this section is to prove an existence theorem for the stochastic system (4.30), (2.12), and (2.23). First, we recall a well-known existence theorem for stochastic equations which we will use in our analysis.

## *6.1 Preliminaries*

Recall the existence theorem for stochastic equations proved in [24, pp. 165-173]. Let $W(t)=W(t,\omega)$ be a $d_1$-dimensional real-valued Wiener process on $(\Omega,\Sigma,m)$, $\Sigma_t\subset\Sigma$ be the increasing filtration (see Sect. 4.3) complete with respect to $\sigma$-algebra $m$-measurable sets $\Sigma_m$ and coordinated with $W(t)$, i.e., $(W(t),\Sigma_t)$ is a Wiener process.

We consider the stochastic equation

$$d\xi(s) = \sigma(s,\xi(s))dW(s) + b(s,\xi(s))ds, \quad s \geqslant t, \quad \text{and} \quad \xi(t) = \xi^0(t), \quad (6.1)$$

where $t \geqslant 0$ is fixed and $\xi^0(t)$ is a $\Sigma_t$-measurable $d$-dimensional vector. The integral form of (6.1) is

$$\xi(s) = \xi^0(t) + \int_t^s \sigma(r,\xi(r))\, dW(r) + \int_t^s b(r,\xi(r))\, dr\,. \quad (6.2)$$

By the solution of (6.2) we mean a $d$-dimensional process $\xi(s) = \xi(s,\omega)$ that is $\Sigma_s$-measurable in $\omega$ for all $s \geqslant t$, is continuous in $s$ and defined for $\omega \in \Omega$ and $s \in (t,\infty)$, and satisfies (6.2) for all $s \in [t,\infty)$ almost everywhere. Additionally, $\sigma(s,x) \in L^{2,\mathrm{loc}}$ (in $s$), $b(s,x) \in L^{1,\mathrm{loc}}$ (in $s$), and they are defined on $\Omega \times (t,\infty)$ for all $x \in \mathbb{R}^d$ and have values in $(d \times d_1)$-matrices and in $\mathbb{R}^d$ correspondingly. We assume that $\sigma$ and $b$ are continuous on $x$ for all $(s,\omega)$ and, for each $T, R \in [0,\infty)$ and $\omega \in \Omega$, the bound

$$\int_0^T \sup_{|x|\leqslant R} \Big[\|\sigma(s,x)\|^2 + |b(s,x)|\Big] ds < \infty \quad (6.3)$$

holds.

**Theorem 6.1 (**see [24, p. 166]). *Let the following conditions hold.*

(i) *Lipschitz condition: For any $R > 0$ there exists a function $K_r(R) > 0$ belonging to $L^{1,\mathrm{loc}}$ as a function of $(\omega, r)$ such that for all $|x|, |y| \leqslant R$, $r > 0$, and $\omega \in \Omega$,*

$$2\big(x-y, b(r,x) - b(r,y)\big) + \|\sigma(r,x) - \sigma(r,y)\|^2 \leqslant K_r(R)|x-y|^2\,. \quad (6.4)$$

(ii) *Growth condition: For all $x \in E^d$, $r > 0$, and $\omega \in \Omega$*

$$2(x, b(r,x)) + \|\sigma(r,x)\|^2 \leqslant K_r(1)(1+|x|^2)\,. \quad (6.5)$$

*Then the stochastic equation* (6.1) *has a solution and any two solutions are identical.*

## *6.2 Bounded approximations*

Theorem 6.1 is not applicable to the problem (4.30), (2.12), and (2.23) because (4.30) has the term $|\psi(t)|^2\psi(t)$ that does not satisfy the growth condition (6.5). Moreover, (4.30) holds for $k \in G_h$; due to the boundary conditions (2.23), the function $k \to \psi_k(\cdot)$ should be defined for $k \in \partial G_h^+$ as well.

The requirement that $\psi_k$ is defined for $kh \in \partial G_h^+$ does not bring any difficulties because it is enough for us to put into (4.30) an expression for $\psi_k$

with $kh \in \partial G_h^+$ given in (2.23) and after that to solve the Cauchy problem (4.30) and (2.12).

Temporarily, we modify (4.30) to an equation that satisfies the conditions of Theorem 6.1. To this end, we introduce the function $\gamma_N \in C^\infty(0,\infty)$ such that

$$\gamma_N(t) = \begin{cases} t, & t \in [0,N], \\ \text{increases monotonically}, & t \in (N, N+1), \\ N+1, & t \geqslant N+1, \end{cases} \tag{6.6}$$

and consider the system

$$\begin{aligned} d\psi_k(t) + \Big\{(i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + \gamma_N(|\psi_k(t)|^2)\psi_k(t)\Big\}dt \\ = \widehat{r}[\psi_k(t)]dW_k(t) \end{aligned} \tag{6.7}$$

instead of (4.30). We consider the problem (6.7) and (2.12). In this problem, the functions $\psi_k(t)$ and $W_k(t)$ are complex-valued. (Recall that $\widehat{r}[\psi_k(t)]W_k(t)$ in (6.7) is understood in the sense of (4.31).) If we introduce the real and imaginary parts of these functions, substitute them into (6.2), and separate the real and imaginary parts of the resulting equations, we obtain a system that satisfies all the conditions of Theorem 6.1. Therefore, the following theorem holds.

**Theorem 6.2.** *The problem* (6.7) *and* (2.12) *has a solution, and any two solutions with identical initial data* (2.12) *are identical.*

We apply to (6.7) the same arguments that were applied to (4.30) that led us to the bound (5.6). Then we obtain the following bounds for the solution $\psi_k(t) \equiv \psi_k^N(t)$ of (6.7) and (2.12):

$$\begin{aligned} \|\boldsymbol{\psi}^N(t)\|^2_{L^{2,h}} + 2\int_0^t \Big[\|(i\nabla_h + \mathbf{A})\psi^N\|^2_{L^{2,h}} \\ + h^d \sum_k \gamma_N\big(|\psi_k^N(s)|^2\big)|\psi_k^N(s)|^2\Big] ds \\ -2\int_0^t h^d \sum_k \operatorname{Re}\Big(\overline{\psi_k^N}\widehat{r}[\psi_k^N]dW_k\Big) \\ = \int_0^t h^d \sum_k \left(2|\psi_k^N|^2 + \sum_j |\Theta_{kj}|^2\mu_j|\widehat{r}[\psi_k^N]|^2\right) ds + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}}\,. \end{aligned} \tag{6.8}$$

This is the analogue of (5.9); after some transformations, we obtain the final inequality

$$E\|\boldsymbol{\psi}^N(t)\|^2_{L^{2,h}} + E\int\limits_0^t \Big( \|\nabla_h^+ \boldsymbol{\psi}^N\|^2_{L^{2,h}} + h^d \sum_k \gamma_N(|\psi_k^N(\tau)|^2)|\psi_k^N(\tau)|^2 \Big) d\tau \leqslant C_2 e^{C_1 t}\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right) . \tag{6.9}$$

## *6.3 Solvability of the discrete stochastic system*

Recall (see [26, p. 54]) that a random variable $\tau = \tau(\omega)$, $\omega \in \Omega$, that takes values in $[0,\infty]$ is called the *stopping* time (relative to $\Sigma_t$) if $\{\omega : \tau(\omega) > t\} \in \Sigma_t$ for every $t \in (0,\infty)$.

Let $M < N$. We introduce the (random) Markov moment

$$\tau_M(\omega) = \begin{cases} \inf\{\tau > 0 \,:\, \|\boldsymbol{\psi}^N(\tau,\omega)\|^2_{L^{2,h}} \geqslant M\} \text{ for } \omega \in \Omega \\ \infty \quad \text{if } \|\boldsymbol{\psi}^N(\tau,\omega)\|_{L^{2,h}} \leqslant M \quad \forall\, \tau > 0 \,. \end{cases} \tag{6.10}$$

Clearly, $\tau_M(\omega)$ is the stopping time. For fixed $t > 0$ we set $t_M = t_M(\omega) = t \wedge \tau_M(\omega)$, which is the stopping time as well.

We substitute $t = t_M(\omega)$ with $M/h^d < N$ into (6.8) and obtain

$$\|\psi_k^N(t_M))\|^2_{L^{2,h}} + 2\int\limits_0^{t_M} \Big( \|(i\nabla_h^+ + A)\boldsymbol{\psi}^N\|^2_{L^{2,h}} + \|\boldsymbol{\psi}^N\|^4_{L^{4,h}} \Big) dt - 2\int\limits_0^{t_M} \sum_k \mathrm{Re}\left( \overline{\psi_k^N} \widehat{r}[\psi_k^N] dW_k \right) = \int\limits_0^{t_M} \sum_k \left(2|\psi_k^N|^2 + \mu_k |\widehat{r}[\psi_k^N]|^2\right) dt + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{6.11}$$

We note that we have changed the term $h^d \sum_k \gamma_N(|\psi_k^N(s)|^2)|\psi_k^N(s)|^2$ to $\|\boldsymbol{\psi}^N(s)\|^4_{L^{2,h}}$ because, for $s < t_M$, $\|\boldsymbol{\psi}^N(s)\|^2_{L^{2,h}} \leqslant M/h^d < N$ and therefore for every $k$, $|\psi_k^N(s)|^2 < N$. This justifies the aforementioned change. Therefore, repeating the derivation of (6.9) from (6.8), we find that (6.11) implies the bound

$$E\|\boldsymbol{\psi}^N(t_M)\|^2_{L^{2,h}} + E\int\limits_0^{t_M} \Big( \|\nabla_h \boldsymbol{\psi}^N(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}^N(s)\|^4_{L^{4,h}} \Big) ds \leqslant C_2 e^{C_1 t}\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right) , \tag{6.12}$$

where $C_1$ and $C$ do not depend on $N$.

Taking into account the definition of $t_M(\omega)$ and the arguments written before (6.12), we see that $\boldsymbol{\psi}^N(s)$ satisfies not only (6.7), but also the equation

$$\begin{aligned}\psi_k^N(t_M) + \int_0^{t_M} \left[(i\nabla_h + A_k)^2\psi_k^N(s) - \psi_k^N(s) + |\psi_k^N(s)|^2\psi_k^N(s)\right] ds \\ = \int_0^{t_M} r_k[\boldsymbol{\psi}^n(s)]dW_k^N(s) + \boldsymbol{\psi}_0 \,.\end{aligned} \tag{6.13}$$

It is clear that for each $N_1$ satisfying $M < N < N_1$ the vector-valued function $\boldsymbol{\psi}^{N_1}(s) = \boldsymbol{\psi}^{N_1}(s,\omega)$ (that evidently exists) satisfies (6.13) as well for almost all $\omega \in \Omega$ and $s \in (0, t_M(\omega))$. This implies that for almost all $\omega \in \Omega$

$$\psi_k^N(s,\omega) = \psi_k^{N_1}(s,\omega) \quad \forall\, kh \in G_h \text{ for } s \in (0, t_M(\omega)) \ . \tag{6.14}$$

Indeed, $\boldsymbol{\psi}^N(s)$, as well as $\boldsymbol{\psi}^{N_1}(s)$, satisfies (6.13) in which the term $|\psi_k^N(s)|^2\psi_k^N(s)$ is changed to $\gamma_{N_1}\left(|\psi_k^N(s)|^2\right)\psi_k^N(s)$. But for this equation, all solutions are indistinguishable.

The equality (6.14) permits us to define the vector-valued function $\boldsymbol{\psi}(s,\omega)$ as follows:

$$\psi_k(s,\omega) = \psi_k^N(s,\omega), \quad kh \in G_h \quad \forall\, N > M/h^d,\ s \in (0, t_M(\omega))\,. \tag{6.15}$$

By virtue of (6.12), the function $\boldsymbol{\psi}(s,\omega)$ defined in (6.15) satisfies

$$\begin{aligned}E\|\boldsymbol{\psi}(t_M)\|^2_{L^{2,h}} + E\int_0^{t_M} \left(\|\nabla_h\boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds \\ \leqslant C_2 e^{C_1 t} E\left(\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right)\end{aligned} \tag{6.16}$$

and the inequality in (6.16) is true since, by definition, $t_M \leqslant t$.

**Lemma 6.3.** *For almost all $\omega \in \Omega$, $t_M \nearrow t$ as $M \to \infty$.*

*Proof.* The definitions of $\tau_M$ and $t_M$ imply that for each $M_1 > M$ the inequalities

$$\tau_M \leqslant \tau_{M_1}, \quad t_M \leqslant t_{M_1} \leqslant t \tag{6.17}$$

hold. Then, by the monotone convergence theorem, there exists $t_\infty(\omega) \leqslant t$ and $\tau_\infty(\omega) \leqslant \infty$ such that $\tau_M(\omega) \to \tau_\infty(\omega)$ and $t_M(\omega) \to t_\infty(\omega) \leqslant t$ as $M \to \infty$ for almost any $\omega \in \Omega$. Suppose that there exists a set $b \subset \Sigma$ satisfying $m(b) > 0$ such that $t_\infty(\omega) < t$ for all $\omega \in b$. This means that for each $M > 0$, $\tau_M(\omega) = t_M(\omega) < t$ for $\omega \in b$ and therefore $\tau_\infty(\omega) = t_\infty(\omega)$, $\omega \in b$. The definition (6.15) of $\psi_k(s,\omega)$ and (6.8) imply that for almost all $\omega \in \Omega$, $\|\boldsymbol{\psi}(s)\|^2_{L^{2,h}}$ is continuous for $s \in (0, \tau_\infty(\omega))$. Due to the continuity for almost all $\omega \in \Omega$, $\tau_M(\omega) < \tau_{M+1}(\omega) < \cdots < \tau_{M+K}(\omega) < \cdots$. Recall that for $\omega \in b$, $\tau_M(\omega) \to \tau_\infty(\omega) < t$. Hence, by (6.10), we obtain

$$\int_b \|\boldsymbol{\psi}(\tau_M(\omega),\omega)\|^2_{L^{2,h}} m(d\omega) \geqslant (M-1)\int_b m(d\omega) \to \infty \quad \text{as } M \to \infty\ . \tag{6.18}$$

Since for $\omega \in b$, $\tau_M(\omega) = t_M(\omega)$, we obtain, by (6.16),

$$\int_b \|\boldsymbol{\psi}(\tau_M(\omega),\omega)\|^2_{L^{2,h}} m(d\omega) \leqslant E\|\boldsymbol{\psi}(t_M)\|^2_{L^{2,h}} \leqslant C_1 e^{Ct}\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} \tag{6.19}$$

for $M \to \infty$. But (6.19) contradicts (6.18) and therefore the proof is complete. □

By Lemma 6.3, (6.10), and the fact that $t_M = t \wedge \tau_M$ for almost all $\omega \in \Omega$ the function

$$G(t_M,\omega) = \|\boldsymbol{\psi}(t_M(\omega),\omega\|^2_{L^{2,h}} + \int_0^{t_M(\omega)} \left(\|\nabla_h^+ \boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds$$

increases monotonically as $M \to \infty$. By (6.16) and the Beppo Levi theorem, the function $G(t,\omega)$ is well-defined for a nonrandom value $t$. Hence,

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + E\int_0^t \left(\|\nabla_h^+ \boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds \leqslant C_2 e^{C_1 t}\|\boldsymbol{\psi}\|^2_{L^{2,h}}\ . \tag{6.20}$$

Therefore, the function $\psi_k(s,\omega)$ defined in (6.15) can be extended up to a function defined for every nonrandom $t > 0$, and this function satisfies (4.32) and is equivalent to (4.30). Uniqueness of the obtained solution of (4.32) follows from (6.15) and the uniqueness of $\psi_k^N(s,w)$. Applying the arguments of Sects. 5.3 and 5.5 to $\psi_k(s,\omega)$, we find that $\psi_k(s,\omega)$ satisfies the estimates (5.13) and (5.52).

Thus, we have proved the following theorem.

**Theorem 6.4.** *There exists a continuous $\Sigma_{h,t}$-adapted random process $\{\boldsymbol{\psi}(t,\omega)\} = \{\psi_k(t,\omega), kh \in G_h\}$ given for $t \geqslant 0$ and such that* (4.32) *holds for all $t \geqslant 0$ with probability one. This process $\boldsymbol{\psi}(t,\omega)$ satisfies the inequalities* (5.6), (5.13), *and* (5.52). *The process $\boldsymbol{\psi}$ that satisfies the aforementioned properties is unique.*

**Definition 6.5.** The random process $\{\boldsymbol{\psi}(t,\omega)\} = \{\psi_k(t,\omega), kh \in G_h\}$ that satisfies all the properties mentioned in Theorem 6.4 is called the *strong solution* of (4.30), (2.12), and (2.23) or (what is equivalent) the strong solution of (4.32).

To prove the solvability of the stochastic problem for the Ginzburg–Landau equation, we need certain additional bounds for the strong solution of (4.32). These bounds will be proved in the next section.

# 7 Smoothness of the Strong Solution with respect to $t$

We establish two estimates for the solution of the problem (4.30), (2.23), and (2.12). Specifically, we estimate the mean maximum and the mean modulus of continuity. In both estimates we follow [44, pp. 352-360].

## 7.1 Estimate of the mean maximum

In this subsection, we present a result for the mean maximum of the solution of the problem (4.30) and (2.12).

**Proposition 7.1.** *Let* $\boldsymbol{\psi}(t)$ *be the strong solution of* (4.30) *and* (2.12). *Then*

$$E(\|\boldsymbol{\psi}(t)\|_{L^\infty(0,T;L^{2,h})}) \leqslant C(T) < \infty \quad \textit{for any } T > 0, \tag{7.1}$$

*where* $C(T)$ *does not depend on* $h$.

*Proof.* We obtain from (5.8) and (5.9) that

$$\begin{aligned}\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + \int_0^t 2\left(\|\boldsymbol{\psi}\|^2_{L^{2,h}} + 1\right)\sum_j \mu_j \, d\tau \\ +2\int_0^t \sum \mathrm{Re}\,(\overline{\psi}_k \widehat{r}[\psi_k] dW_k)\end{aligned} \tag{7.2}$$

and from this estimate, along with the Gronwall inequality, we obtain

$$\begin{aligned}\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} e^{2t\sum\limits_j \mu_j} + C\int_0^t e^{2\sum\limits_j \mu_j(t-\tau)}\big(\mathrm{Re}\sum_k (\overline{\psi}_k \widehat{r}[\psi_k] dW_k)(\tau) \\ + \sum_j \mu_j \tau\big)\mu_j \, d\tau \,.\end{aligned} \tag{7.3}$$

Multiplying both sides of (7.3) by $e^{-2t\sum\limits_j \mu_j}$ and taking the maximum over $t \in [0, T]$, we obtain

$$\sup_{t\in[0,T]} \left( e^{-2t\sum\limits_j \mu_j} \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \right) \leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + C_2 + \sup_{t\in[0,T]} \|\mathbf{M}\| , \tag{7.4}$$

where $\mathbf{M} = (M_k(t), kh \in G_h)$, and

$$M_k(t) = C \int\limits_0^t e^{-2\tau\sum\limits_j \mu_j} \mathrm{Re}\left(\overline{\psi_k(\tau)} r[\psi_k]\right) dW_k(\tau) .$$

The process $M_k(t)$ is a martingale with respect to the flirtation $\Sigma_t$ (see [44, p. 353]). This, due to the Birkholder-Gaudi inequality, implies

$$E \sup_{[0,T]} |M_k(t)| \leqslant [C(T)]^{\frac{1}{2}} ;$$

see [44, p. 353]. Therefore, taking the mathematical expectation of both sides of (7.4), we obtain (7.1). □

Similarly to Proposition 7.1, using (5.56), (5.67), and (5.68) (instead of (5.8) and (5.9)), one can prove Proposition 7.1'. Let $\psi(t)$ be the strong solution of (4.29) and (2.12). Then

$$E\|\nabla_h^+ \psi\|_{L^\infty(0,T;L^{2,h})} \leqslant C(T) < \infty \quad \text{for any } T > 0 , \tag{7.5}$$

where $C(T)$ does not depend on $h$.

## *7.2 Estimate of the auxiliary random process*

We introduce the seminorm

$$\|\boldsymbol{\psi}\|_{C^\alpha_{T,h}} = \sup_{\substack{0\leqslant t_1 < t_2 \leqslant T \\ |t_1 - t_2| \leqslant 1}} \frac{\|\boldsymbol{\psi}(t_1) - \boldsymbol{\psi}(t_2)\|_{L^{2,h}}}{|t_1 - t_2|^\alpha} \quad \forall\, T > 0 . \tag{7.6}$$

Recall that the function $r(\lambda)$ is defined in (3.19).

Now define

$$S(\lambda) = \int\limits_0^\lambda \frac{d\mu}{r(\mu)} . \tag{7.7}$$

In accordance with the general definition (3.20) and (3.21), we denote

$$\begin{aligned} S[\psi_k(t)] &= S(\mathrm{Re}\,\psi_k(t)) + iS(\mathrm{Im}\,\psi_k(t)) \\ \widehat{S}[\psi_k] z &= S(\mathrm{Re}\,\psi_k) \mathrm{Re}\, z + iS(\mathrm{Im}\,(\psi_k)) \mathrm{Im}\, z \end{aligned} \tag{7.8}$$

for each complex number $z$. Applying the Ito formula to $S[\psi(t)]$, i.e., applying the Ito formula to the function $S(\operatorname{Re}\psi_k)$ and to the function $S(\operatorname{Im}\psi_k)$, we obtain

$$\begin{aligned} dS[\psi_k(t)] = -\widehat{S}'[\psi_k(t)]\Big((i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + |\psi_k|^2\psi_k\Big)dt \\ +\widehat{S}'[\psi_k]\widehat{r}[\psi_k]dW_k + \tfrac{1}{2}\widehat{S}''[\psi_k](\widehat{r^2}[\psi_k][dW_k]^2)\,. \end{aligned} \tag{7.9}$$

Here,

$$\begin{aligned} &\widehat{S}'[\psi_k]\widehat{r}[\psi_k]dW_k \\ &\quad = S'(\operatorname{Re}\psi_k)r(\operatorname{Re}\psi_k)d\operatorname{Re}W_k + iS'(\operatorname{Im}\psi_k)r(\operatorname{Im}\psi_k)d\operatorname{Im}W_k = dW_k \end{aligned} \tag{7.10}$$

and the last equality holds because of (7.7). Note that the first term on the right-hand side of (7.9) should be understood in the same sense as was indicated in the second relation of (7.8). Moreover, by virtue of (4.39) and (7.7),

$$\begin{aligned} \widehat{S}''[\psi_k]\widehat{r}[\psi_k]\widehat{r}[\psi_k](d\operatorname{Re}W_k)^2 &= \frac{1}{2}S''(\operatorname{Re}\psi_k)r^2(\operatorname{Re}\psi_k)(d\operatorname{Re}W_k)^2 \\ &\quad + \frac{i}{2}S''(\operatorname{Im}\psi_k)r^2(\operatorname{Im}\psi_k)(d\operatorname{Im}W_k)^2 \\ &= -\frac{1}{2}\Big(r'(\operatorname{Re}\psi_k) + ir'(\operatorname{Im}\psi_k)\Big)\sum_{jh\in G_h}|\Theta_{kj}|^2\mu_j dt \\ &= -\frac{1}{2}\sum_j |\Theta_{kj}|^2\mu_j r'[\psi_k]dt\,. \end{aligned} \tag{7.11}$$

As a result, we obtain from (7.9)–(7.11) and (7.7) that

$$\begin{aligned} dS[\psi_k(t)] = \Big\{&-\widehat{r^{-1}}[\psi_k]\big((i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k\big) \\ &-\frac{1}{2}r'[\psi_k]\sum_j|\Theta_{kj}|^2\mu_j\Big\}dt + dW_k\,, \end{aligned} \tag{7.12}$$

where the equality is understood in the sense of (3.20) and (3.21). Now using the results from [44], we derive an estimate for $\|S(\boldsymbol{\psi})\|_{C^{\alpha}_{T,h}}$.

Denote $\mathbf{Z}(t)$ as

$$\mathbf{Z}(t) = S[\boldsymbol{\psi}(t)] - \mathbf{W}(t)\,. \tag{7.13}$$

Equalities (7.12) and (7.13) imply

$$\dot{\mathbf{Z}}_k(t) = \frac{d}{dt} Z_k(t) = -\Big(\widehat{r^{-1}}[\psi_k]\big((i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k\big) - r'[\psi_k]\sum_j |\Theta_{kj}|^2 \mu_j\Big) . \tag{7.14}$$

**Lemma 7.2.** *For any $T > 0$ the inequality*

$$\|\mathbf{Z}\|_{C^{\frac{1}{2}}_{T,h}} \leqslant C\Big\{1 + \Big(\int_0^T \big(\|\Delta_h \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}}\big) dt\Big)^{1/2}\Big\} \tag{7.15}$$

*holds, where $C$ does not depend on $h$.*

*Proof.* By virtue of (7.14) and (3.19), we have

$$\|\dot{\mathbf{Z}}(t)\|^2_{L^{2,h}} \leqslant C_1 \left(\|\Delta_h \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}\|^6_{L^{6,h}} + 1\right)$$

and therefore

$$\|\dot{\mathbf{Z}}(t)\|_{L^{2,h}} \leqslant C_1^{1/2}\big(\|\Delta_h \boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^3_{L^{6,h}} + 1\big) . \tag{7.16}$$

This inequality implies

$$\begin{aligned}
\|\mathbf{Z}(t_2) - \mathbf{Z}(t_1)\|_{L^{2,h}} &\leqslant \int_{t_1}^{t_2} \|\dot{\mathbf{Z}}(t)\|_{L^{2,h}} dt \\
&\leqslant C_1^{1/2} \int_{t_1}^{t_2} \Big(\|\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\Delta_h \boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\boldsymbol{\psi}\|^3_{L^{6,h}} + 1\Big) dt \\
&\leqslant C\Big[\Big(\int_{t_1}^{t_2} \big[\|\Delta_h \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}}\big] dt\Big)^{\frac{1}{2}} (t_2 - t_1)^{1/2} + (t_2 - t_1)\Big] .
\end{aligned}$$

By using the definition (7.6), we obtain the desired result (7.15). □

Recall that the Levi modulus is the function $\aleph(t) = |t \ln t|^{1/2}$ and the norm $\|\mathbf{W}\|_{C_{L,T,h}}$ is defined as

$$\|\mathbf{W}\|_{C_{L,T,h}} = \sup_{\substack{0 \leqslant t_1 < t_2 < T \\ |t_1 - t_2| < 1/e}} \frac{\|\mathbf{W}(t_1) - \mathbf{W}(t_2)\|_{L^{2,h}}}{\aleph(t_2 - t_1)} . \tag{7.17}$$

Recall that $\Lambda_h = P_h^* \Lambda$ is the distribution of the Wiener process $\mathbf{W}(t)$ from (4.11), where $\Lambda$ is the distribution of the initial Wiener process (see (3.1)). The measure $\Lambda_h$ is defined on $\mathcal{B}(\mathbb{C}_h)$, where $\mathbb{C}_h = C(0, \infty; L^{2,h}(G_h))$. In [44, p. 356] the following assertion was proved.

**Lemma 7.3.** *There exist positive constants $C_1$ and $C_2$ independent of $h$ (and of $\Lambda_h$) such that for any $\alpha > 0$*

$$\Lambda_{T,\alpha}^h \equiv \Lambda_h(\{\mathbf{W} \in \mathbb{C}_h : \|\mathbf{W}\|_{C_{L,T}} > C_1 \alpha\}) \leqslant C_2 T \frac{\sqrt{Tr_h}}{\alpha} 2^{-\alpha^2/2\, Tr_h} , \tag{7.18}$$

*where $Tr_h = \sum\limits_{jh \in G_h} \mu_j$ is the trace of the correlation operator $\widehat{\mathcal{K}}$ defined in* (4.17) *and below* (4.18) *and corresponding to the Wiener process $\mathbf{W}(t)$.*

**Lemma 7.4.** *The process $S[\boldsymbol{\psi}(t)]$ with function $S$ defined in* (7.7) *satisfies the bound*

$$\begin{aligned} \|S[\boldsymbol{\psi}]\|^2_{C_{L,T,h}} \leqslant 2C_1 \Big[ 1 + \int_0^T \Big( &\|\Delta_h \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} \\ &+ \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}} \Big) dt \Big] + 2\|\mathbf{W}\|^2_{C_{L,T,h}} , \end{aligned} \tag{7.19}$$

*where $C_1$ does not depend on $h$.*

*Proof.* The bound (7.19) directly follows from (7.13) and Lemma 7.2 if we take into account that $|t_2 - t_1|^{1/2} \leqslant \aleph(t_2 - t_1) \equiv \big|(t_2 - t_1) \ln |t_2 - t_1|\big|^{\frac{1}{2}}$ for $|t_2 - t_1| < \frac{1}{e}$. □

**Theorem 7.5.** *Let $\boldsymbol{\psi}(t)$ be the solution of the stochastic problem* (4.30), (2.23)*, and* (2.12)*. Then the bound*

$$E\|S[\boldsymbol{\psi}]\|^2_{C_{L,T,h}} \leqslant C(T) , \tag{7.20}$$

*holds, where $C(T)$ does not depend on $h$.*

*Proof.* We take the mathematical expectation of both sides of (7.19) and, to estimate the right-hand side, we use Lemma 7.3 and the bounds (5.6), (5.13), and (5.52). As a result, we obtain (7.20). □

## 7.3 Estimate of the mean modulus of continuity

We define the norm

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} = \sup_{\substack{0\leqslant t_1<t_2<T \\ |t_1-t_2|<1/e}} \frac{\|\boldsymbol{\psi}(t_1)-\boldsymbol{\psi}(t_2)\|_{L^{1,h}}}{\aleph(t_1-t_2)}\,, \tag{7.21}$$

where

$$\|\boldsymbol{\psi}\|_{L^{1,h}} = h^d \sum_{kh\in G_h} |\psi_k|\,. \tag{7.22}$$

Note that by virtue of definitions (7.7) and (3.19), the function $S(\lambda)$ possesses the inverse function $R(S)$

$$R(S(\lambda)) = \lambda\,. \tag{7.23}$$

**Lemma 7.6.** *There exists a constant* $C>0$ *such that*

$$|\lambda_1-\lambda_2| \leqslant C(1+|\lambda_1|+|\lambda_2|)\ |S(\lambda_1)-S(\lambda_2)| \quad \forall\, \lambda_1,\lambda_2\in\mathbb{R}^1\,. \tag{7.24}$$

*Proof.* Let $\lambda_1>\lambda_2$. Then $S(\lambda_1)>S(\lambda_2)$. By virtue of (7.23) and (7.7), $R'(S(\lambda)) = r(\lambda)>0$. Therefore, using the Lagrange theorem, we obtain

$$\begin{aligned}\lambda_1-\lambda_2 &= R(S(\lambda_1))-R(S(\lambda_2)) \\ &\leqslant \sup_{\mu\in[\lambda_2,\lambda_1]} R'(S(\mu))|S(\lambda_1)-S(\lambda_2)| \leqslant R'(S(\lambda_1))\ |S(\lambda_1)-S(\lambda_2)| \\ &\leqslant C(1+|\lambda_1|+|\lambda_2|)\ |S(\lambda_1)-S(\lambda_2)|\,.\end{aligned} \tag{7.25}$$

□

**Theorem 7.7.** *Let* $\boldsymbol{\psi}(t)$ *be the strong solution of the stochastic problem* (4.30), (2.23), *and* (2.12). *Then the following estimate holds*:

$$E\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} \leqslant C(T)\,. \tag{7.26}$$

*Proof.* It is enough to prove the bound

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} \leqslant C\left(1+\sup_{0\leqslant t\leqslant T}\|\boldsymbol{\psi}(t)\|^2 + \|S(\boldsymbol{\psi})\|^2_{C_{L,T,h}}\right) \tag{7.27}$$

because, after taking the mathematical expectation of both sides of (7.27) and using (7.20) and (7.1), we obtain (7.26). Substituting $\lambda_i = \operatorname{Re}\psi_k(t_i)$, $i=1,2$, or $\lambda_i = \operatorname{Im}\psi_k(t_i)$, $i=1,2$, into (7.25) gives

$$h^d \sum_{kh\in G_h} |\psi_k(t_1)-\psi_k(t_2)|$$

$$\leqslant C(1 + \|\boldsymbol{\psi}(t_1)\|_{L^{2,h}} + \|\boldsymbol{\psi}(t_2)\|_{L^{2,h}})\|S[\boldsymbol{\psi}(t_1)] - S[\boldsymbol{\psi}(t_2)]\|_{L^{2,h}} .$$

Dividing both parts of this bound by the Levi modulus and taking into account the definitions (7.17) and (7.21), we obtain

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} \leqslant C(1 + \sup_{t\in[0,T]} \|\boldsymbol{\psi}(t)\|_{L^{2,h}})\|S[\boldsymbol{\psi}(t)]\|_{C_{L,T,h}} .$$

This inequality clearly implies (7.27). □

# 8 Compactness Theorems

In order to pass to the limit in the stochastic equation (4.30), we need some compactness theorems which we present in this section.

## *8.1 On compact sets in $L^2(G)$*

For almost all $\omega \in \Omega$ the strong solution $\boldsymbol{\psi}(t)$ of Equation (4.30) belongs to $L^2(0,T;L^{2,h}(G_h))$, where $L^{2,h}(G_h) = P_h L^2(G)$ is the space defined before (4.8). Let $1 \leqslant p < \infty$. Similarly to the space $L^{2,h}(G_h)$, we can introduce the space $L^{p,h}(G_h)$ of vector-valued functions $\boldsymbol{\psi} = \{\psi_k : kh \in G_h\}$ supplied with the norm

$$\|\boldsymbol{\psi}\|^p_{L^{p,h}(G_h)} = h^d \sum_{kh\in G_h} |\psi_k|^p . \tag{8.1}$$

Clearly, $L^{p,h}(G_h) = P_h L^p(G)$, where the operator $P_h$ is defined as well as the operator $P_h$ from (4.9). As in (4.10), one can prove that the operator $P_h : L^p(G) \to L^{p,h}(G_h)$ is bounded. We define the space

$$H^1_{A,h}(G_h) = \{\boldsymbol{\psi} \in L^{2,h}(G_h),\ \psi \text{ is defined on } \partial G_h^+ \text{ by (2.23)}\} \tag{8.2}$$

and the norm (see (5.5)):

$$\begin{aligned} \|\boldsymbol{\psi}\|^2_{H^1_{A,h}} &= h^d \widetilde{\sum_{j,k}} (|\partial^+_{j,h}\psi_k|^2 + |\psi_k|^2) \\ &\equiv h^d \sum_{j=1}^d \sum_{kh\in G_h\cup\partial G_h^+(-j)} (|\partial^+_{j,h}\psi_k|^2 + |\psi_k|^2) . \end{aligned} \tag{8.3}$$

We can identify the space $L^{p,h}(G_h)$ (as well as the space (8.2) ) with subspaces of functions belonging to $L^p(G)$ by the operator (4.14):

$$L^{p,h} \ni \boldsymbol{\psi} = \{\psi_k\} \to \psi_h(x) = \sum_{kh\in G_h} h^{-d}\psi_k \mathcal{X}_{Q_k}(x) \in L^p(G) , \tag{8.4}$$

where $\mathcal{X}_{Q_k}(x)$ is the characteristic function of the set $Q_k$ (i.e., $\mathcal{X}_{Q_k}(x) = 1$, for $x \in Q_k$, $\mathcal{X}_{Q_k}(x) = 0$ for $x \neq Q_k$) and the sets $Q_k$ are defined by (4.1)–(4.5). We denote by $\widehat{L}^{p,h}(G)$ the subspace of $L^p(G)$ formed by identifying (8.4). The following assertion follows from (4.6)–(4.7) and a bound similar to (4.10).

**Proposition 8.1.** *The spaces $L^{p,h}(G_h)$ and $\widehat{L}^{p,h}(G)$ are isomorphic* (*so the norm* (8.1) *is equivalent to the norm of $\widehat{L}^{p,h}(G) \subset L^p(G)$*) *and the isomorphism is defined by* (8.4).

In the space $\widehat{L}^{2,h}(G)$, the norm (8.3) generates the norm

$$\|\psi_h\|^2_{\widehat{H}^1_{A,h}} = \int\limits_G \left( \sum_{j=1}^{d} \frac{|\psi_h(x+e_jh) - \psi_h(x)|^2}{h^2} + |\psi_h(x)|^2 \right) dx \,. \tag{8.5}$$

To calculate the finite difference in (8.5), we assume that $\psi_h(x)$ is defined on $\bigcup\limits_{kh \in G_h \cup \partial G_h^+} Q_k$ and, on sets $Q_k$, $kh \in G_h^+$, $\psi_h(x)$ is defined with the help of (2.23).

More precisely, in order to determine the finite difference quotient $(\psi_h(x+e_jh) - \psi_h(x))/h$, we use the polyhedra

$$Q_k = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \,:\, x_j \in [h(k_j - \frac{1}{2})), h(k_j + \frac{1}{2})), j = 1, \dots, d\} \tag{8.6}$$

for each $kh \in G_h \cup \partial G_h^+$, defining $\psi_h(x)$ for $x \in Q_k$ with $kh \in \partial G_h^+(\pm m)$ by (2.23). Then for $kh \in \partial G_h^-(\pm m)$ we change the polyhedra (8.6) in the definition of the quotient $(\psi_h(x+e_mh)-\psi_h(x))/h$ on the appropriate set $Q_k$ from (4.2)–(4.5).

We denote by $\widehat{H}^1_{A,h}(G) = P_h^* H^1_{A,h}(G_h)$, where $P_h^*$ is the operator (4.14) and $H^1_{A,h}(G_h)$ is the space (8.2) with the norm given by (8.3). Similar to Proposition 8.1, the following assertion holds.

**Proposition 8.2.** *The operator* (4.14) *establishes an isomorphism between $H^1_{A,h}(G_h)$ and $\widehat{H}^1_{A,h}(G)$, i.e., the norms* (8.3) *and* (8.5) *are equivalent with constants independent of $h$.*

*Proof.* One can easily obtain the necessary estimates with the help of the explanation near (8.6) and the relations (4.6) and (4.7). □

Below, we assume that $h = h_n = 2^{-n}h_0 \to 0$ as $n \to \infty$. For each $R > 0$ we set

$$B_R(\widehat{H}^1_{A,h}) = \{\psi \in \widehat{L}^{2,h}(G) \,:\, \|\psi\|_{\widehat{H}^1_{A,h}} \leqslant R\} \tag{8.7}$$

and

$$B_R(H^1) = \{\psi \in H^1(G) \,:\, \|\psi\|_1 \leqslant R\} \,. \tag{8.8}$$

**Lemma 8.3.** *For each $R > 0$ the set*

$$\Theta_R \equiv \bigcup_{n=1}^{\infty} B_R(\widehat{H}^1_{A,h_n}) \cup B_R(H^1) \tag{8.9}$$

*is compact in $L^2(G)$ and in $L^1(G)$.*

*Proof.* We choose from an arbitrary sequence $\psi_m \in \Theta_R$ a subsequence converging in $L^2(G)$. Two cases are possible: (i) there exists $n_0 > 0$ such that $\psi_m \in \bigcup_{n=1}^{n_0} B_R(\widehat{H}^1_{A,h_n}) \cup B_R(H^1)$ for each $m$; (ii) there exists a subsequence $\{m'\}$ of the sequence $\{m\}$ such that $\psi_{m'} \in B_R(\widehat{H}^1_{A,h_{n_{m'}}})$ and $n_{m'} \to \infty$ as $m' \to \infty$.

In the first case, we can choose a subsequence $\{m'\} \subset \{m\}$ such that (a) $\psi_{m'} \in \bigcup_{n=1}^{n_0} B_R(\widehat{H}^1_{A,h_n})$ for all $m' \in \{m'\}$ or (b) $\psi_{m'} \in B_R(H^1)$ for all $m' \in \{m'\}$. For case (a), we can choose a converging subsequence $\{\psi_{m''}\}$ from $\{\psi_{m'}\}$ because $\bigcup_{h=1}^{n_0} B_R(\widehat{H}^1_{A,h_n})$ is a finite dimensional closed bounded set. For case (b), we can choose a converging subsequence $\{\psi_{m''}\}$ because, as is well known, the embedding $H^1(G) \subset L_2(G)$ is compact.

In the second case, we can choose a subsequence $\{\psi_{m''}\} \subset \{\psi_{m'}\}$ weakly converging to $\widetilde{\psi}(x)$ in $L^2(G)$ as $m'' \to \infty$. Moreover, by virtue of the definitions (8.7) –(8.9), for each $\varepsilon$ there exists $\delta > 0$ and $N > 0$ such that for all $h$ satisfying $\|h\| < \delta$ and for all $n \geqslant N$,

$$\int |\psi_n(x) - \psi_n(x-h)|^2 < \varepsilon\,. \tag{8.10}$$

Then, by (8.10), we use standard arguments to choose a subsequence $\{\psi_q\} \subset \{\psi_m\}$ such that $\|\psi_q - \widetilde{\psi}\|_{L^2(G)} \to 0$ as $q \to \infty$ (see [40, Chapt. 1, Sect. 4]. □

## *8.2 Compact sets in the space of time-dependent functions*

Let $E_0$, $E$, and $E_1$ denote reflexive Banach spaces such that the embeddings $E_0 \subset E \subset E_1$ are continuous and the embedding $E_0 \subset E$ is compact. Then the Dubinsky theorem (see [44, p. 131-132]) can be stated as follows.

**Theorem 8.4.** *Let $1 < q, q_1 < \infty$, and let $M$ be a bounded set in $L_q(0,T;E_0)$ consisting of functions $u(t)$ equicontinuous in $C(0,T;E_1)$. Then $M$ is relatively compact in $L_{q_1}(0,T;E)$ and $C(0,T;E_1)$.*

We establish some variants of this theorem which we will need. First, let us apply this theorem to the following situation. We introduce the space

$$\mathcal{W} = \{\psi(t,x) \in L^2(0,T;H^1(G)) \cap C^L(0,T;L^1(G))\}\,, \tag{8.11}$$

where

$$C^L(0,T;L^1(G)) = \Big\{\psi(t,x),(t,x) \in (0,T)\times G\ :\ \|\psi\|_{C_{L,T,1}} = \sup_{\substack{0\leqslant t_1<t_2\leqslant T\\ |t_1-t_2|<e^{-1}}} \frac{\|\psi(t_1,\cdot)-\psi(t_2,\cdot)\|_{L^1(G)}}{\aleph(t_2-t_1)} + \sup_{0\leqslant t\leqslant T}\|\psi(t,\cdot)\|_{L^1(G)} < \infty\Big\}, \tag{8.12}$$

where again $\aleph(t) = |t\ln t|^{\frac{1}{2}}$ for $t>0$.

**Theorem 8.5.** *The set*

$$B_R(\mathcal{W}) = \{\psi(t,x) \in \mathcal{W}\ :\ \|\psi\|_{\mathcal{W}} \leqslant R\} \tag{8.13}$$

*is compact in the space* $L^4((0,T)\times G) \cap C(0,T;L^1(G))$.

*Proof.* To apply Theorem 8.4, we take $E_0 = H^1(G)$, $E = L^4(G)$, $E_1 = L^1(G)$, and $M = B_R(\mathcal{W})$. Clearly, $M$ consists of functions that are equicontinuous in $C(0,T;E_1)$. □

Let

$$\mathcal{W}_h = \Big\{\psi(t,x) \in L^2(0,T;\widehat{H}^1_{A,h}(G))\ :\ \|\psi\|_{C_{L,T,1}} = \sup_{\substack{0\leqslant t_1<t_2<T\\ |t_1-t_2|<e^{-1}}} \frac{\|\psi(t_1,\cdot)-\psi(t_2,\cdot)\|_{L^1(G)}}{\aleph(t_2-t_1)} + \sup_{0\leqslant t\leqslant T}\|\psi(t,\cdot)\|_{L^1(G)} < \infty\Big\} \tag{8.14}$$

and

$$B_R(\mathcal{W}_h) = \{\psi \in \mathcal{W}_h\ :\ \|\psi\|_{C_{L,T,1}} + \|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h}(G))} \leqslant R\}\,. \tag{8.15}$$

Since $\mathcal{W}_h$ consists of functions equicontinuous in $C(0,T;L^1(G))$, the following assertion holds.

**Proposition 8.6.** *The set* (8.15) *is compact in the space* $L^4((0,T)\times G) \cap C(0,T;L^1(G))$.

The following theorem then holds.

**Theorem 8.7.** *For each* $R>0$ *the set*

$$\Theta_R = \bigcup_{n=1}^{\infty} B_R(\mathcal{W}_{h_n}) \cup B_R(\mathcal{W}) \tag{8.16}$$

*is compact in* $Z_T \equiv L^2((0,T)\times G)\cap C(0,T;L^1(G))$*. Here,* $h_n = h_0 2^{-n}$ *and* $B_R(\mathcal{W}_h)$ *and* $B_R(\mathcal{W})$ *are the sets* (8.15) *and* (8.13) *respectively.*

*Proof.* By virtue of Theorem 8.5 and Proposition 8.6, the sets $B_R(\mathcal{W}_{h_n})$ and $B_R(\mathcal{W})$ are compact in $L^4((0,T)\times G)$. Now, to complete the theorem, we apply the proof sketched in Lemma 8.3. □

# 9 Weak Solution of the Discrete Stochastic Problem

Our aim here is to pass to the limit as $h\to 0$ in the problem (4.30), (2.12), and (2.23) in order to prove an existence theorem for the boundary value problem (3.22), (2.2), and (2.3) for the stochastic Ginzburg–Landau equation. For this purpose, we need the definition of a weak solution of (4.30), (2.12), and (2.23).

## *9.1 Definition of the weak solution for the discrete problem*

Recall that we suppose that the initial condition from (2.3) is a random process, i.e., $\psi_0(x)=\psi_0(x,\omega)$ , $x\in G$, $\omega\in\Omega$, and we suppose that the map $\psi_0 : \Omega\to L^2(G)$ is measurable, i.e., $\psi_0 : \Sigma\to\mathcal{B}(L^2(G))$ where $(\Omega,\Sigma,m)$ is the initial probability space. Moreover, we assume that the random value $\psi_0$ and the Wiener process $W(t,x,\omega)$ defined in Sect. 3 are independent, i.e., for each $B\in\mathcal{B}(C(0,\infty;L^2(G))$ and $b\in\mathcal{B}(L^2(G))$,

$$m\big(\{\omega : W(\cdot,\cdot,\omega)\in B, \psi_0(\cdot,\omega)\in b\}\big) \\ = m\big(\{\omega : W(\cdot,\cdot,\omega)\in B\}\big)m\big(\{\omega : \psi_0(\cdot,\omega)\in b\}\big)\,. \tag{9.1}$$

Now we construct certain projections of $\psi_0(\cdot,\omega)$ and $W(\cdot,\cdot,\omega)$. Using the projection $P_h : L^2(G)\to L^{2,h}(G_h)$ defined in (4.9), we can define the projection $P_h\psi_0(\omega)$ and $P_hW(t,\omega)$ defined on $(\Omega,\Sigma,m)$ and taking the values $P_h\psi_0(\omega)\in L^{2,h}(G_h)$ and $P_hW(t,\omega)\in C(0,\infty;\widehat{L}^{2,h}(G))$ respectively. Moreover, using the projection $P_h^* : L^{2,h}(G_h)\to\widehat{L}^{2,h}(G)\subset L^2(G)$ defined in (4.14), we can define the projections $P_h^*P_h\psi((\cdot,\omega))$, $\omega\in\Omega$, with values belonging to $C(0,\infty;\widehat{L}^{2,h}(G))\subset C(0,\infty;L^2(G))$. So, using the notation

$$\widehat{P}_h = P_h^* P_h\,, \tag{9.2}$$

where $P_h$ is the operator (4.9) and $P_h^*$ is the operator (4.14), we define the random value

$$\Omega\ni\omega\to\widehat{P}_h\psi_0(\cdot,\omega)\in\widehat{L}^{2,h}(G)\subset L^2(G) \tag{9.3}$$

and the Wiener random process

$$\Omega \ni \omega \to (\widehat{P}_h W)(\cdot,\cdot,\omega) \in C(0,\infty;\widehat{L}^{2,h}(G)) \subset C(0,\infty;L^2(G))\,. \tag{9.4}$$

The relationship (9.1) for $\psi_0(\cdot,\omega)$ and $W(\cdot,\cdot,\omega)$ implies the independence of $\widehat{P}_h(\psi_0(\cdot,\omega))$ and $\widehat{P}_h W(\cdot,\cdot,\omega)$.

Note that the increasing filtration $\Sigma_t$ corresponding to the Wiener process $W(t,x,\omega)$ corresponds to the Wiener process $\widehat{P}_h W(t,x,\omega)$ as well.

We define the space of functions

$$\mathcal{U}_h = L^{2,\mathrm{loc}}(0,\infty;\widehat{H}^1_{A,h}(G)) \cap C^L(0,\infty;L^1(G)) \cap L^{6,\mathrm{loc}}(0,\infty;L^6(G))\,, \tag{9.5}$$

where the index $L$ means the Levi modulus $|t\ln t|^{1/2}$ for $t \in (0,1/e)$. It is clear that $\mathcal{U}_h$ is a Frechet space with seminorms

$$\|\psi\|_{\mathcal{U}_{h,T}} = \|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h}(G))} + \|\psi\|_{C^L(0,T;L^1(G))} + \|\psi\|_{L^6((0,T)\times G)}\,. \tag{9.6}$$

With the aid of the solution $\boldsymbol{\psi}(t,\omega)$ of the problem (4.30) and (2.12), we can define the random process

$$\Omega \ni \omega \to (P_h^*\boldsymbol{\psi})(\cdot,\cdot,\omega) \equiv \psi_h(\cdot,\cdot,\omega) \in \mathcal{U}_h\,. \tag{9.7}$$

The space $\mathcal{U}_h$ from (9.5) is well connected with the solution $\psi_h$ but we will need also in the following a more extensive separable Frechet space for the solution; we have

$$Z = L^{2,\mathrm{loc}}(0,\infty;L^2(G)) \cap C(0,\infty;L^1(G)) \tag{9.8}$$

with finite seminorms given by

$$\|\psi\|_{Z_T} \equiv \|\psi\|_{L^2(0,T;L^2(G))} + \|\psi\|_{C(0,T;L^1(G))}\,, \quad T > 0. \tag{9.9}$$

We will also use the spaces

$$\begin{aligned} Z_T &= L^2(0,T;L^2(G)) \cap C(0,T;L^1(G)),\\ \mathcal{U}_{h,T} &= L^2(0,T;\widehat{H}^1_{A,h}(G)) \cap C^L(0,T;L^1(G)) \cap L^6((0,T)\times G) \end{aligned} \tag{9.10}$$

supplied with the norms (9.9) and (9.6) correspondingly.

Recall that $B(Z)$ is a Borel $\sigma$-algebra of the space $Z$ and $B_{\mathcal{U}_h}(Z) = B(Z)\cap \mathcal{U}_h$. By virtue of Theorem 2.1 from [44, Chapt. 2], $B_{\mathcal{U}_h}(Z) \subset \mathcal{B}(\mathcal{U}_h)$.

**Definition 9.1.** The weak statistical solution of (4.30), (2.12), and (2.23) is the probability distribution of the random process (9.7), i.e.,

$$\nu_h(B) = m\big(\{\omega\,:\,\psi_h(\cdot,\cdot,\omega) \in B\}\big) \quad \forall\, B \in \mathcal{B}_{\mathcal{U}_h}(Z)\,. \tag{9.11}$$

## *9.2 The equation for the weak solution of the discrete problem*

Taking the integral form of the Ito equation (7.12) and applying the operator $P_h^*$ from (4.14) we obtain

$$
\begin{aligned}
L_h(\psi_h) &\equiv S[\psi_h(t,x)] - S[\psi_{h,0}(\cdot)] \\
&+ \int_0^t \Big\{ \widehat{r^{-1}}[\psi_h(\tau,x)] \Big( (i\nabla_h + \widehat{P}_h A(x))^2 \psi_h(\tau,x) - \psi_h + |\psi_h|^2\psi_h \Big) \\
&- \frac{1}{2}\widehat{r'}[\psi_h] \sum_{j,k} |\Theta_{kj}|^2 \mu_j \mathcal{X}_{Q_k}(x) V(Q_k)^{-1} \Big\} d\tau = \widehat{P}_h W(t,x)\,.
\end{aligned}
\tag{9.12}
$$

Let $\gamma_0$ be the restriction operator of functions $f(t,\cdot)$ at $t=0$, i.e., $\gamma_0 f = f(0,\cdot)$. We consider the operator

$$\mathfrak{A}_h \equiv (\gamma_0, L_h) \,:\, \mathcal{U}_h \to L^1(\Omega) \times Z\,, \tag{9.13}$$

where $L_h$ is the operator given in (9.12).

**Proposition 9.2.** *The operator* (9.13) *is continuous.*

*Proof.* The proof of this assertion is obvious because the space $\widehat{H}^1_{A,h}(G)$ forming the space $\mathcal{U}_h$ is finite dimensional. □

We want to use the operator (9.13) to rewrite the weak solution (9.11) in some other form. Recall that the full preimage of the set $B \times B_0$, where $B \in Z$, $B_0 \in L^1(G)$, is defined as follows:

$$\mathfrak{A}_h^{-1}(B_0 \times B) = \{\psi \in \mathcal{U}_h \,:\, \mathfrak{A}_h \psi = (\gamma_0\psi, L_h\psi) \in B_0 \times B\}\,. \tag{9.14}$$

By virtue of Proposition 9.2, $\mathfrak{A}_h^{-1}(B_0 \times B) \in \mathcal{B}(\mathcal{U}_h)$. This full preimage is strictly connected to the solution $\psi_h(t,x)$ of the problem (9.12). Indeed, we have

$$
\begin{aligned}
\psi_h(t,x,\omega) &= \quad \psi_h(t,x,\psi_0(\cdot,\omega), W(\tau \in (0,t),\cdot,\omega)) \\
&= \mathfrak{A}_h^{-1}(t,x,\psi_0(\cdot,\omega), W(\tau \in (0,t),\cdot,\omega))\,,
\end{aligned}
\tag{9.15}
$$

where, in contrast to (9.14), $\mathfrak{A}_h^{-1}$ is the inverse (i.e., uniquely valued) operator of the operator $\mathfrak{A}_h$. The domain of the operator (9.15) is the set of initial conditions and right-hand sides, where the solution of (4.30), (2.12), and (2.23) exists and is unique and therefore the solution of (9.12) possesses the same property. This domain is given by

$$\mathcal{D}(\mathfrak{A}_h^{-1}) = (\widehat{P}_h L^1(G), \widehat{P}_h \widehat{W})\,, \tag{9.16}$$

where $\widehat{W}$ is the image of the Wiener process defined in Sect. 3:

$$\widehat{W} = \{W(\cdot,\cdot,\omega), \omega \in \Omega\}, \quad W(\cdot,\cdot,\omega) \text{ is a Wiener process.} \tag{9.17}$$

Definition (9.17) implies that

$$\widehat{W} \text{ is a } \Lambda\text{-measurable set}\,. \tag{9.18}$$

Now for each $B_0 \in \mathcal{B}(L^1(G))$ and $B \in \mathcal{B}(Z)$, we can write (see [44, p. 343])

$$\begin{aligned}(\mathfrak{A}_h^* \nu_h)(B_0 \times B) &= \nu_h(\mathfrak{A}_h^{-1}(B_0 \times B)) \\ &= \nu_h(\{\psi_h \in \mathcal{U}_h \,:\, \mathfrak{A}_h \psi_h \in \widehat{P}_h B_0 \times \widehat{P}_h B\}) \\ &= m\big(\{\omega \,:\, \widehat{P}_h \psi_0(\cdot,\omega) \in \widehat{P}_h B_0, \widehat{P}_h W(\cdot,\cdot,\omega) \in \widehat{P}_h B\}\big) \\ &= \widehat{P}_h^* \mu(B_0) \times \widehat{P}_h^* \Lambda(B) = \mu_h(B_0)\Lambda_h(B)\,. \end{aligned} \tag{9.19}$$

The relation

$$(\mathfrak{A}_h^* \nu_h)(B_0 \times B) = \mu_h(B_0)\Lambda_h(B) \quad \forall\, B_0 \in \mathcal{B}(L^1(G)), B \in \mathcal{B}(Z) \tag{9.20}$$

is the desired equation for the weak statistical solution $\nu_h$ defined in (9.11).

## 10 Passage to the Limit in a Family of $\nu_{h_n}$

To take this limit, we need certain additional compactness results which we present here.

### *10.1 Compactness of the family of measures $\nu_{h_n}$*

Recall that $h_n = h_0 2^{-n}$. First, we establish some estimates for $\nu_{h_n}$ We denote by $\Gamma_T$ the restriction operator on the interval $(0,T)$, i.e.,

$$\Gamma_T \psi = \psi|_{(0,T)}\,. \tag{10.1}$$

Let $Z_T = \Gamma_T Z$ and

$$\nu_{hT}(C) = \nu_h(\Gamma_T^{-1} C) \quad \forall\, C \in \mathcal{B}(\mathcal{U}_T)\,. \tag{10.2}$$

**Theorem 10.1.** *Suppose that the distribution $\mu(d\psi_0)$ of the initial condition $\psi_0(x,\omega)$ satisfies the inequality*

$$\int \big(\|\psi_0\|^2_{L^2(G)} + \|\nabla\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\big)\mu(d\psi_0) < \infty\,. \tag{10.3}$$

*Then the measure $\nu_{hT}$ satisfies the estimates*

$$\int\limits_{Z_T} \Big(\|\psi(t,\cdot)\|^2_{L^2(G)} + \int\limits_0^t (\|\nabla_h^+\psi(t,\cdot)\|^2_{L^2(G)} + \|\psi(t,\cdot)\|^4_{L^4(G)})dt\Big)\nu_{hT}(d\psi)$$

$$\leqslant C_1 e^{CT}\Big(1 + \int\limits_{L^2(G)} \|\psi_0\|^2\mu(d\psi_0)\Big)\,, \tag{10.4}$$

$$\int\limits_{Z_T} \Big(\|\psi(t,\cdot)\|^4_{L^4(G)} + \int\limits_0^t \|\psi(t,\cdot)\|^6_{L^6(G)})d\tau\Big)\,\nu_{hT}(d\psi)$$

$$\leqslant C_2 e^{Ct}(1 + \int\limits_{L^2(G)} \|\psi_0\|^4\mu(d\psi_0)\,, \tag{10.5}$$

*and*

$$\int\limits_{Z_T} \Big(\|\psi\|^2_{L^\infty((0,T);L^2(G))} + \|\psi\|_{C_{L,T,1}}\Big)\,\nu_{hT}dt \leqslant C(T)\,, \tag{10.6}$$

*where the constants $C_1$, $C_2$, and $C$ do not depend on $h$ and $T$ and $C(T)$ does not depend on $h$.*

*Proof.* From the usual definition (10.2) and (9.11) of the measure $\nu_{hT}$ and Propositions 8.1 and 8.2, we can immediately derive (10.4) from (5.6), (10.5) from (5.13), and (10.6) from the bounds given in (7.1) and (7.26). □

Our goal is to prove the weak compactness of the measures $\nu_{h_n}$. For this purpose, we use the following well-known theorem which is proved, for example, in [19] .

**Theorem 10.2 (**Prokhorov). *A family $\mathcal{M}$ of measures defined on the Borel $\sigma$-algebra $\mathcal{B}(Z)$ of a separable Banach space $Z$ is weakly compact if*

(a) $\sup\{\mu(Z)\,:\,\mu\in\mathcal{M}\} < \infty$,

(b) *for any* $\varepsilon > 0$ *there exists a compact set* $K \subset Z$ *such that* $\sup\{\mu(Z\backslash K)\,:\,\mu\in\mathcal{M}\} < \varepsilon$.

**Lemma 10.3.** *The set of measures $\nu_{h_nT}$, $n\in\mathbb{N}$, is weakly compact on $Z_T = L^2((0,T)\times G)\cap C(0,T;L^1(G))$.*

*Proof.* We use Theorem 10.2. Since $\nu_{h_n T}$ are probability measures, the condition (a) of the Prokhorov theorem is satisfied. We must check condition (b) of the theorem. For a compact set $K$ we take the set $\Theta_R$ introduced in (8.16). By Theorem 8.7, $\Theta_R$ is compact in $Z_T$. Note that the measure $\nu_{h_k T}$ is concentrated in $L^2(0,T;\widehat{L}^{2,h_k}(G))$ and therefore

$$\operatorname{supp}\nu_{h_k}\cap\Theta_R = B_R(W_{h_k})\cap\operatorname{supp}\nu_{h_k T}\,. \tag{10.7}$$

Therefore, using (10.7) and the Chebyshev inequality as well as the bounds (10.4)–(10.6), we obtain

$$\begin{aligned}\int\limits_{L^2(0,T\times G)\setminus\Theta_R}\nu_{h_k T}(d\psi) &= \int\limits_{L^2(0,T;\widehat{L}^{2,h_k}(G))\setminus B_R(W_{h_k})}\nu_{h_k T}(d\psi)\\ &\leqslant \frac{1}{R}\int\Big(\|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h_k}(G))}+\|\psi\|_{C_{L,T,1}}\Big)\nu_{h_k,T}(d\psi)\leqslant\frac{C}{R}\,,\end{aligned} \tag{10.8}$$

where $C$ does not depend on $k$. The inequality (10.8) implies that the measure $\nu_{h_m}$ satisfies condition (b). Therefore, the assertion of the lemma follows from Prokhorov's theorem. □

## 10.2 Passage to the limit

In this section, we demonstrate that the set of measures $\nu_{h_n}$, $n\in\mathbb{N}$, is weakly compact on $Z$ and thus we can choose a subsequence that converges weakly to $\nu$ in $Z$.

**Theorem 10.4.** *The set of measures $\nu_{h_n}$, $n\in\mathbb{N}$, is weakly compact on $Z$.*

*Proof.* The proof is similar to the proof given in [44, p. 361]. □

By virtue of Theorem 10.4, we can choose from the sequence of measures $\{\nu_{h_n}\}$ the subsequences $\{\nu_{h_j}\}$ that converges weakly to $\nu$ on $Z$, i.e.,

$$\nu_{h_j}\to\nu \quad \text{as } j\to\infty \text{ weakly on } \mathcal{Z}. \tag{10.9}$$

We will show that the measure $\nu$ is the weak solution (see Definition 12.1 below) of the stochastic problem (3.22), (2.2), and (2.3).

# 11 Estimates for the Weak Solution

We first prove an estimate for $\nu_h$.

## 11.1 An estimate for $\nu_h$

In order to prove the analogue of the estimate given in (5.52), we have to define the second finite difference $\Delta_h\psi_h(x)$ for $\psi_h(x) \in \widehat{L}^{2,h}(G)$.

Assuming that the lattice function $\boldsymbol{\psi} = \{\psi_k\}$ satisfies (2.23), we can then define the norm

$$\|\boldsymbol{\psi}\|^2_{H^2_{A,h}(G_h)} = h^d \sum_{kh\in G_h} \left(|\Delta_h\psi_k|^2 + |\nabla_h^+\psi_k|^2 + |\psi_k|^2\right). \tag{11.1}$$

We set

$$H^2_{A,h}(G_h) = \{\boldsymbol{\psi} \in L^{2,h}(G_h),\quad \boldsymbol{\psi} \text{ satisfies (2.23), supplied with the norm (11.1) }\}. \tag{11.2}$$

We also define the space $\widehat{H}^2_{A,h}(G)$ along with its norm as

$$\begin{aligned} &\widehat{H}^2_{A,h}(G) = P_h^* H^2_{A,h}(G_h)\,, \\ &\|\psi_h\|^2_{\widehat{H}^2_{A,h}} = \int\limits_G \Big(|\Delta_h\psi_h(x)|^2 + |\nabla_h^+\psi_h(x)|^2 + |\psi_h(x)|^2\Big)dx\,. \end{aligned} \tag{11.3}$$

Note that in a neighborhood of $\partial G$, the finite difference $|\Delta_h\psi_h(x)|^2$ is calculated as was explained near (8.5). More precisely, to calculate the difference $|\Delta_h\psi_h(x)|^2$, we use the polyhedra $Q_k$ from (8.6) and, after these calculations, we change these polyhedra in a neighborhood of $\partial\Omega$ on appropriate polyhedra; see (4.2)–(4.5). The value of $\psi_h(x)$ on this polyhedra $Q_k$ is defined by (2.23).

The following assertion which is analogous to Propositions 8.1 and 8.2 can be proved .

**Proposition 11.1.** *The spaces* $H^2_{A,h}(G_h)$ *and* $\widehat{H}^2_{A,h}(G)$ *are isomorphic and the norms in* (11.3) *and* (11.1) *are equivalent.*

The following theorem easily results from the estimate (5.52).

**Theorem 11.2.** *The measure* $\nu_{hT}$ *satisfies the estimate*

$$\begin{aligned} &\int\limits_{Z_T} \Big(\|\nabla_h^+\psi(t)\|^2_{L^2(G)} + \int\limits_0^t \|\Delta_h\psi(\tau,\cdot)\|^2_{L^2(G)}\,d\tau\Big)\nu_{h,T}(d\psi) \\ &\qquad \leqslant C_1 e^{Ct}\Big(1 + \int \left(\|\nabla_h^+\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\right)\mu(d\psi_0)\Big)\,. \end{aligned} \tag{11.4}$$

Recall that $h_n = 2^{-n}h_0$. Below we will need modifications of Theorems 10.1 and 11.2, where on the left-hand sides of the inequalities in these theorems we need to replace $\nabla_h^+$ and $\Delta_h$ with $\nabla_{h_m}^+$ and $\Delta_{h_m}$ respectively. In addition, $\nu_{h,T}(d\psi)$ must be changed to $\nu_{h_n,T}(d\psi)$ for $n > m$. To establish such estimates, we prove some preliminary lemmas in the next section.

## *11.2 Preliminary lemmas*

In this section, we provide several preliminary results which will be needed to prove estimates for the measure $\nu$.

**Lemma 11.3.** *Let $u_k$, $k = 1, \dots, N$, $h > 0$, be a lattice function. Then*

$$\sum_{k=1}^{N-n} \left| \frac{u_{k+n} - u_k}{nh} \right|^2 \leqslant \sum_{k=1}^{N} \left| \frac{u_{k+1} - u_k}{h} \right|^2 . \tag{11.5}$$

*Proof.* Since $(a_1 + \cdots + a_j)^2 \leqslant j(a_1^2 + \cdots + a_j^2)$ for positive $a_1, \dots, a_j$, we have

$$\begin{aligned} \sum_{k=1}^{N-n} \left| \frac{u_{k+n} - u_k}{nh} \right|^2 &= \frac{1}{n^2} \sum_{k=1}^{N-n} \left| \sum_{j=1}^{n} \frac{u_{k+j} - u_{k+j-1}}{h} \right|^2 \\ &\leqslant \frac{1}{n} \sum_{k=1}^{N-n} \sum_{j=1}^{n} \left| \frac{u_{k+j} - u_{k+j-1}}{h} \right|^2 \\ &\leqslant \sum_{k=1}^{N} \left| \frac{u_{k+1} - u_k}{h} \right|^2 , \end{aligned}$$

where to obtain the last inequality we have taken into account that the previous sum can be represented as the sum of groups of identical summands and the number of identical summands in each group are not more than $n$. □

**Lemma 11.4.** *Let $u_k$, $k = 0, \dots, N$, $h > 0$, be a lattice function. Then*

$$\sum_{k=n}^{N-n} \left| \frac{u_{k+n} - 2u_k + u_{k-n}}{(nh)^2} \right|^2 \leqslant 4 \sum_{k=1}^{N-1} \left| \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} \right|^2 . \tag{11.6}$$

*Proof.* For $k = 1, \dots, N-1$ we set $\Delta_h u_k = u_{k+1} - 2u_k + u_{k-1}$. One can prove that

$$u_{k+n} - 2u_k + u_{k-n} = \sum_{j=1}^{n} j \Delta_h u_{k+n-j} + \sum_{j=1}^{n-1} (n-j) \Delta_h u_{k-j} .$$

Therefore,

$$\sum_{k=n}^{N-n}\left|\frac{u_{k+n}-2u_k+u_{k-n}}{(nh)^2}\right|^2$$

$$\leqslant \frac{2n}{(nh)^4}\sum_{k=n}^{N-n}\Big(\sum_{j=1}^{n} j^2\left|\Delta_h u_{k+n-j}\right|^2+\sum_{j=1}^{n-1}(n-j)^2\left|\Delta_h u_{k-j}\right|^2\Big)$$

$$\leqslant \frac{2}{nh^4}\sum_{k=n}^{N-n}\Big(\sum_{j=1}^{n}\left|\Delta_h u_{k+n-j}\right|^2+\sum_{j=1}^{n-1}\left|\Delta_h u_{k-j}\right|^2\Big)\ \leqslant\ 4\sum_{k=1}^{N-1}\left|\frac{\Delta_h u_k}{h^2}\right|^2$$

because the maximal number of elements in each group of identical summands in the penultimate sum is $2n$. □

For the approximate domain $G_{h_n}\cup\partial G^+_{h_n}$ we intend to define the first and second finite difference quotients $\nabla^+_{h_m}$ and $\Delta_{h_m}$ with $m<n$. For $j=1,\dots,d$ denote

$$G_{h_n}(+j;h_m)=\left\{k\in\mathbb{Z}^d\ :\ kh_n\in G_{h_n},(k+2^{n-m}e_j)h_n\in G_{h_n}\cup\partial G^+_{h_n}\right\}.$$

Clearly, for each $kh_n\in G_{h_n}(+j;h_m)$, the difference quotient $\partial^+_{j,h_m}u_k=(u_{k+2^{n-m}e_j}-u_k)/h_m$ is well defined. In an analogous manner, we denote

$$G_{h_n}(-j;h_m)=\left\{k\in\mathbb{Z}^d\ :\ kh_n\in G_{h_n},(k-2^{n-m}e_j)h_n\in G_{h_n}\cup\partial G^+_{h_n}\right\}.$$

Let

$$G_{h_n}(+;h_m)=\bigcap_{j=1}^{d}G_{h_n}(+j;h_m),\qquad G_{h_n}(-;h_m)=\bigcap_{j=1}^{d}G_{h_n}(-j;h_m)\quad(11.7)$$

and

$$G_{h_n}(h_m)=\bigcap_{j=1}^{d}\left(G_{h_n}(+j;h_m)\cap G_{h_n}(-j,h_m)\right).\qquad(11.8)$$

It is clear that the subsets (11.7) and (11.8) of $G_{h_n}\cap\partial G^+_{h_n}$ satisfy the following properties: for all $kh_n\in G_{h_n}(+;h_m)$, the operator $\nabla^+_{h_m}u_k$ is well defined and, for $kh_n\in G_{h_n}(h_m)$, the operator $\Delta_{h_m}u_k$ is well defined.

We are now in a position to prove the following lemma.

**Lemma 11.5.** *For each* $\boldsymbol{\psi}\in L^{2,h_n}(G_{h_n})$

$$h_n^d\sum_{kh_n\in G_{h_n}(+;h_m)}\left|\nabla^+_{h_m}\psi_k\right|^2\leqslant\|\nabla^+_{h_n}\boldsymbol{\psi}\|^2_{L^{2,h_n}(G_{h_n})}\qquad(11.9)$$

*and*

$$h_n^d \sum_{kh_n \in G_{h_n}(h_m)} |\Delta_{h_m}\psi_k|^2 \leqslant 4\|\Delta_{h_n}\boldsymbol{\psi}\|^2_{L^{2,h_n}(G_{h_n})}\,. \tag{11.10}$$

*Proof.* The bound (11.9) is a direct corollary of Lemma 11.3 and the bound (11.10) follows directly from Lemma 11.4. □

Denote

$$G(h_m) = \bigcup_{kh_n \in G_{h_n}(h_m)} Q_k\,, \tag{11.11}$$

where the sets $Q_k$ are defined by (4.1) with $h = h_n$. Then, using the operator $P^*_{h_n}$ defined in (4.14), we immediately obtain from Lemma 11.5 the following assertion.

**Lemma 11.6.** *For each* $\psi(x) \in \widehat{H}^2_{A,h_n}(G)$

$$\int\limits_{G(h_m)} \left|\nabla^+_{h_m}\psi(x)\right|^2 dx \leqslant C \int\limits_G \left|\nabla^+_{h_n}\psi(x)\right|^2 dx \tag{11.12}$$

*and*

$$\int\limits_{G(h_n)} |\Delta_{h_m}\psi(x)|^2\, dx \leqslant C \int\limits_G |\Delta_{h_n}\psi(x)|^2\, dx\,, \tag{11.13}$$

*Recall that calculation of the functions from (11.12) and (11.13) near the boundaries of* $G$ *and* $G(h_m)$ *should be made as was explained near (8.5) and (11.3) with* $h = h_n$*. where* $C$ *does not depend on* $\psi$*,* $n$*, or* $m$*.*

At last we are now able to prove the following corollary of Proposition 7.1 and Theorems 10.1 and 11.2.

**Theorem 11.7.** *Let the distribution* $\mu(d\psi_0)$ *of the initial condition* $\psi_0(x,\omega)$ *satisfy* (10.3)*. Then for each* $m < n$ *the measures* $\nu_{h_n,T}(d\psi)$ *satisfy the estimates*

$$\begin{aligned}\int\limits_{Z_T}\Big(\int\limits_0^T \big(\|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))} + \|\nabla^+_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}\big)\, d\tau\Big)\nu_{h_nT}(d\psi)\\ \leqslant C_T\Big(1 + \int\limits_{L^2(G)} \big(\|\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)} + \|\nabla\psi_0\|^2_{L^2(G)}\big)\mu_0(d\psi_0)\,,\end{aligned} \tag{11.14}$$

*where the constant* $C_T$ *depends only on* $T$*. Moreover,*

$$\int\limits_{Z_T} \sup_{t\in(0,T)} \|\nabla^+_{h_m}\psi(t,\cdot)\|_{L^2(G(h_m))}\nu_{h_n,T}(d\psi) \leqslant C(T) < \infty \quad \forall T > 0\,, \tag{11.15}$$

*where the constant* $C(T)$ *does not depend on* $h_m$ *or* $h_{n,T}$*.*

*Proof.* The theorem follows immediately from Lemma 11.6, Proposition 7.1, and Theorems 10.1 and 11.2. □

## 11.3 Estimates for the measure $\nu$

We are now in a position to prove the main theorem of this section. We set

$$H^1_\Delta(G) = \Big\{ u(x) \in H^1(G) \,:\, \Delta u(x) \in L^2(G), \\ \|u\|^2_{H^1_\Delta(G)} = \int_G \big(|\Delta u|^2 + |\nabla u|^2 + |u|^2\big) dx < \infty \Big\}. \tag{11.16}$$

**Theorem 11.8.** *Let the distribution $\mu(d\psi_0)$ of the initial condition $\psi_0$ satisfy* (10.3)*. Then the statistical solution $\nu$ constructed in* (10.9) *is supported on the space*

$$supp\,\nu \subset L^{2,\mathrm{loc}}(0,\infty; H^1_\Delta(G)) \cap L^{6,\mathrm{loc}}(0,\infty; L^6(G)) \cap C^L(0,\infty; L^1(G))\,. \tag{11.17}$$

*Moreover, the following estimates hold. For every $T > 0$ there exists a constant $C_T$ depending only on $T$ such that*

$$\int_{\mathcal{U}_T} \Big( \int_0^T \|\Delta\psi\|^2_{L^2(G)} + \|\nabla\psi\|^2_{L^2(G)} + \|\psi\|^6_{L^6(G)} d\tau \Big) \nu_T(d\psi) \\ \leqslant C_T \Big[ 1 + \int_{L^2(G)} \big(\|\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)} + \|\nabla\psi_0\|^2_{L^2(G)}\big) \mu(d\psi_0) \Big] \tag{11.18}$$

$$\int_{\mathcal{U}_T} (\|\psi\|^2_{L^\infty(0,T;L^2(G))} + \|\nabla\psi\|^2_{L^\infty(0,T;L^2(G))}) \nu_T(d\psi) \leqslant C(T) < \infty \quad \forall T > 0 \tag{11.19}$$

*and*

$$\int_{Z_T} \|\psi\|_{C^L(0,T;L^1(G))} \nu(d\psi) \leqslant C(T) < \infty \quad \forall T > 0\,. \tag{11.20}$$

*Proof.* Let $\phi_R(\lambda) \in C^\infty(\mathbb{R}_+)$, $\phi_R(\lambda) = \lambda$ for $\lambda < R$, and $\phi_R(\lambda) = R+1$ for $\lambda \geqslant R+1$. Then the bound (11.14) implies the inequality

$$\int \phi_R \Big( \int_0^T \big(\|\nabla^+_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))} + \|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}$$

$$+\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_{h_nT}(d\psi)\ \leqslant\ \widehat{C}_T$$

$$\equiv C_T(1+\int\limits_{L^2(G)}\big(\|\psi_0\|^2_{L^2(G)}+\|\psi_0\|^4_{L^4(G)}+\|\nabla\psi_0\|^2_{L^2(G)}\big)\mu(d\psi_0)\,. \tag{11.21}$$

Since the functional under the integral on the left-hand side of (11.21) is bounded and continuous on the space $Z$ from (9.8), we can pass to the limit as $n\to\infty$ in (11.21). As a result, we obtain

$$\int\phi_R\Big(\int\limits_0^T\big(\|\nabla^+_{h_m}\phi(\tau,\cdot)\|^2_{L^2(G(h_m))}+\|\varDelta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}$$
$$+\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_T(d\psi)\leqslant\widehat{C}_T\,. \tag{11.22}$$

Using the Beppo Levi theorem, we can pass to the limit in (11.22) as $R\to\infty$ to obtain

$$\int\Big(\int\limits_0^T\big(\|\nabla^+_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}+\|\varDelta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}$$
$$+\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_T(d\psi)\leqslant\widehat{C}_T\,. \tag{11.23}$$

It is easy to prove that

$$\|\varDelta_{h_m}u\|_{L^2(G(h_m))}\to\|\varDelta u\|_{L^2(G)}<\infty$$
$$\|\nabla^+_{h_m}u\|_{L^2(G(h_m))}\to\|\nabla u\|_{L^2(G)}<\infty \tag{11.24}$$

as $h_m\to 0$ if and only if $u\in H^1_\varDelta(G)$. Passing to the limit in (11.23) as $h_m\to 0$, with the help of the Fatou theorem and taking into account (11.24), we find that the measure $\nu_T(d\psi)$ satisfies the inequality

$$\int\int\limits_0^T\|\psi(\tau,\cdot)\|^2_{H^1_\varDelta(G)}d\tau\ \nu_T(du)\leqslant\widehat{C}_T \tag{11.25}$$

and therefore it is supported on the space $L^2(0,T;H^1_\varDelta(G))$. Since the embeddings $H^1_\varDelta(G)\subset H^1(G)\subset L^6(G)$ are continuous when the dimension of $G=d\leqslant 3$, the norm $\|u\|_{L^6}$ is continuous on $H^1_\varDelta(G)$. Therefore, using as the above function $\phi_R(\lambda)$, we can pass to the limit as $n\to\infty$ in the term of the inequality (10.5) containing $\|\psi\|^6_{L^6(G)}$. As a result, we obtain

$$\int \int_0^T \|\psi(\tau,\cdot)\|_{L^6(G)}^6 d\tau \; \nu_T(d\psi) \leqslant \widehat{C}_T \,. \tag{11.26}$$

The inequality (11.19), as well as the bound (11.20) can be obtained with the help of the method used in [44, p. 363]. □

# 12 The Equation for the Weak Solution of the Stochastic Ginzburg–Landau Problem

Roughly speaking, the weak solution is a measure satisfying a certain equation. We begin with the formal derivation of this equation.

## *12.1 Definition of the weak solution*

The stochastic Ginzburg–Landau equation can be written as the Ito differential equation (3.22) with boundary and initial conditions (2.2) and (2.3) respectively. We let $dW(t,x)$ denote the white noise corresponding to the Wiener process defined in Sect. 3.1, $\psi_0(x) = \psi_0(x,\omega) \in L^4(G) \cap H^1(G)$ is a random initial condition with distribution $\mu(d\psi_0)$, and $\psi_0(x)$ and $W(t,x)$ are independent. Let $S(\lambda)$ be the function given in (7.7). Applying formally the Ito formula to the function $S(\psi(t,x))$ and writing the resulting Ito differential in integral form, we obtain

$$\begin{aligned} L(\psi) \equiv\; & S[\psi(t,x)] - S[\psi_0(x)] \\ & + \int_0^t \Big( \widehat{r^{-1}}[\psi(\tau,x)] \left\{ (i\nabla + A(x))^2 \psi(\tau,x) - \psi(\tau,x) + |\psi|^2 \psi(\tau,x) \right\} \\ & \qquad + \frac{1}{2} \widehat{r'}[\psi] \mathcal{K}_{11}(x,x) \Big) \, d\tau = W(t,x) \,, \end{aligned} \tag{12.1}$$

where $\mathcal{K}_{11}(x,x)$ is defined in (3.14).

We introduce the spaces

$$\mathcal{U}_T = L^2(0,T; H_A^2(G)) \cap C^L(0,T; L^1(G)) \cap L^6((0,T) \times G), \quad T > 0 \,, \tag{12.2}$$

and

$$\mathcal{U} = L^{2,\mathrm{loc}}(0,\infty; H_A^2(G)) \cap C^L(0,\infty; L^1(G)) \cap L^{6,\mathrm{loc}}(0,\infty; L^6(G)) \tag{12.3}$$

with the norm for space (12.2)

$$\|\psi\|_{\mathcal{U},T} = \|\psi\|_{L^2(0,T;H^2_A(G))} + \|\psi\|_{C^L(0,T:L^1(G))} \tag{12.4}$$

and with the topology for the space (12.3) defined by the seminorms (12.4) with arbitrary $T > 0$.

Similarly, we consider the continuous operator

$$\mathfrak{A} = (\gamma_0, L) \, : \mathcal{U} \to L^1(G) \times Z \,. \tag{12.5}$$

Repeating formally the derivation of the equation for the weak statistical solution of the approximation for the Ginzburg–Landau equation, we obtain the following analogue of (9.20):

$$(\mathfrak{A}^*\nu)(B_0 \times B) = \mu(B_0)\Lambda(B) \quad \forall \, B_0 \in \mathcal{B}(L^1(G)), \; B \in \mathcal{B}(Z) \,. \tag{12.6}$$

**Definition 12.1.** The probability measure $\nu$ on $\mathcal{B}(\mathcal{U})$ is called the weak statistical solution of the stochastic Ginzburg–Landau equation (3.22) if it is concentrated on $\mathcal{U}$, satisfies the inequalities (11.18), (11.19), and (11.20), and satisfies (12.6), where $\mathfrak{A}$ is the operator from (12.5) and (12.1).

## *12.2 The first steps of the proof for $\nu$ to satisfy* (12.6)

We will show that the measure $\nu$ defined in (10.9) satisfies (12.6). Since the other properties in Definition 12.1 are already proven for $\nu$, this gives that $\nu$ is a weak statistical solution of the stochastic Ginzburg–Landau equation. We can show that (12.6) is equivalent to the equality

$$\int \eta(\gamma_0\psi)\phi(L(\psi))\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0) \int \phi(W)\Lambda(dW) \tag{12.7}$$

for all $\eta \in C_b(L^2(G))$ and $\phi \in C_b(C(0,\infty;L^1(G))$ (recall that $C_b(H)$ is the space of bounded, continuous functions on the Banach space $H$) in the same way as the analogous assertion was proved in [44, p. 364].

We already proved that there exists a strong stochastic solution of the problem (9.12). Therefore, (9.12) implies (9.20) and (9.20) implies that

$$E(\eta(\gamma_0\psi_h)\phi(L_h(\psi_h)) = \int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \int \phi(\widehat{P_h}W)\Lambda(dW) \,, \tag{12.8}$$

where $\widehat{P_h}$ is the operator defined in (9.2). Performing a change of variables on the left-hand side of (12.8), we obtain

$$\int \eta(\gamma_0\widehat{P_h}\psi)\phi(L_h(\widehat{P_h}\psi)\nu_h(d\psi)$$

$$= \int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \int \phi(\widehat{P_h}W)\Lambda(dW)\,. \qquad (12.9)$$

We derive (12.7) by passing to the limit in (12.9) as $h = h_j \to 0$.

Since for each $\psi_0 \in L^2(G)$ and $W \in C(0,\infty; L^1(G))$ we have $\widehat{P_h}\psi_0 \to \psi_0$ as $h \to 0$ in $L^2(G)$ and $\widehat{P_h}W \to W$ as $h \to 0$ in $C(0,\infty; L^1(G))$, we have the following formulas:

$$\begin{aligned} &\int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \to \int \eta(\psi_0)\mu(d\psi_0) \\ &\int \phi(\widehat{P_h}W)\Lambda(dW) \to \int \phi(W)\Lambda(dW) \end{aligned} \qquad (12.10)$$

as $h \to 0$.

We now pass to the limit on the left-hand side of (12.9). By virtue of the arguments in [44, p. 364], it is enough to prove (12.7) only for cylindrical functionals $\phi$, i.e., for $\phi$ that actually depend only on a finite number of arguments and is constant with respect an infinite part of the arguments. But each such functional $\phi(u)$ can be approximated by a finite sum of the form

$$\phi(u) \approx \sum_k e^{i[u,v_k]},$$

where

$$[u, v_k] = \int_0^\infty \int_G u\overline{v}_k dxdt.$$

Consequently, we can modify $\phi(L_h(\widehat{P_h}\psi))$ in (12.9) using $e^{i[L_h(\widehat{P_h}\psi),v]}$. We can now write

$$\int \eta(\gamma_0\widehat{P_h}\psi)\phi(L_h(\widehat{P_h}\psi))\nu_h(d\psi) \rightsquigarrow \int \eta(\gamma_0\widehat{P_h}\psi)e^{i[L_h(\widehat{P_h}\psi),v]}\nu_h(d\psi)\,. \qquad (12.11)$$

We pass to the limit as $h \to 0$ on the right-hand side of (12.11).

Taking $v \in L^2(0,\infty; H^2(G))$, $v(t,x) = 0$ for $t > t_v$, where $H^2(G)$ is the usual Sobolev space, we can rewrite (9.12) as follows:

$$[L_h(\psi), v] = f_{1,h}(\psi) + f_{2,h}(\psi) + f_{3,h}(\psi) \quad \text{with } \widehat{P_h}\psi \text{ changed on } \psi\,, \qquad (12.12)$$

where

$$f_{1,h}(\psi) \;=\; \int_0^\infty \int_G \Big\{ S(\psi(t,x) - S(\gamma_0\widehat{P_h}\psi(\cdot,x))$$

$$+\int_0^t \left(\widehat{r^{-1}}[\psi(\tau,x)]|\psi|^2\psi(\tau,x)-\psi(\tau,x)\right)\,d\tau\Big\}\overline{v(t,x)}\,dxdt\,, \quad (12.13)$$

$$f_{2,h}(\psi)=\frac{1}{2}\int_0^\infty\int_G\int_0^t r'[\psi(\tau,x)]$$
$$\Big(\sum_{kh,jh\in G_h}\mathcal{X}_{Q_j}(x)V(Q_k)^{-1}|\Theta_{jk}|^2\mu_k\Big)d\tau\,\overline{v(t,x)}\,dxdt\,, \quad (12.14)$$

and

$$f_{3,h}(\psi)=$$
$$\int_0^\infty\int_G\int_0^t \widehat{r^{-1}}[\psi(\tau,x)]\Big((i\nabla_h+\widehat{P_h}A(x))^2\psi(\tau,x)\Big)\overline{v(t,x)}\,d\tau dxdt\,, \quad (12.15)$$

where recall that $\widehat{r^{-1}}[\psi(\tau,x)]z$, $z\in\mathbb{C}$, is understood in the meaning of (3.20) and (3.21). First of all, we rewrite $f_{3,h}(\psi)$ by summing by parts. We suppose that each $v(x)\in H^2(G)$ is extended onto $G(\varepsilon)=\{x\in\mathbb{R}^d:\rho(x,G)=\inf_{y\in G}|x-y|<\varepsilon\}$, where $\varepsilon>0$ is fixed, by a fixed extension operator $\mathcal{E}:H^2(G)\to H^2(G(\varepsilon))$ and we denote this extension $\mathcal{E}v(x)$ by $v(x)$. Thus, for small enough $h$, the difference quotients $\partial^+_{h_j}v(x)=\frac{1}{h}(v(x+e_jh)-v(x))$, $j=1,\dots,d$, are well defined for almost all $x\in G$.

**Lemma 12.2.** *The expression* (12.15) *is equivalent to*

$$\begin{aligned}f_{3,h}(\psi)&\\ =\int_0^\infty\int_G\int_0^t&\Big\{\widehat{r^{-1}}[\psi(\tau,x)]\big((\nabla_h^+-i\widehat{P_h}A(x)\big)\psi(\tau,x))\overline{\nabla_h^+v(t,x)}\\ &+\widehat{r^{-1}}[\psi(\tau,x)]\big((i\nabla_h^++\widehat{P_h}A(x))\psi(\tau,x)\big)\overline{\widehat{P_h}A(x)v(t,x)}\\ &+\sum_{j=1}^d\big(\partial^-_{h_j}\widehat{r^{-1}}[\psi(\tau,x)]\big)\\ &\quad\Big(\nabla_h^+-i\widehat{P_h}A(x-he_j)\psi(\tau,x-he_j)\Big)\overline{v(t,x)}\Big\}\,d\tau dxdt\end{aligned} \quad (12.16)$$

*for each* $\psi(\tau,x)=\widehat{P_h}\psi(\tau,x)\in L^2(0,\infty;\widehat{H}^2_{A,h}(G)$ *with the space* $\widehat{H}^2_{A,h}(G)$ *defined in* (11.3)*,* $v(t,x)\in L^2(0,\infty;H^2(G(\varepsilon))$*, and* $v(t,x)=0$ *for* $t>t_0$*.*

*Proof.* We denote

$$\phi(\tau,x) = (\nabla_h^+ - i\widehat{P}_h A(x))\psi(\tau,x)$$

$$\equiv \{\partial_{h_j}^+ - i\widehat{P}_h A^j(x))\psi(\tau,x), j=1,\dots,d\} = \{\phi^j(\tau,x), j=1,\dots,d\}$$

and rewrite (12.15) as

$$f_{3,h}(\psi) = -\int_0^\infty\int_G\int_0^t \widehat{r^{-1}}[\psi(\tau,x)] \Big(\sum_{j=1}^d (\partial_{h_j}^- - i\widehat{P}_h A^j(x))\phi^j(\tau,x)\Big)\overline{v(t,x)}\, d\tau dx dt\,. \tag{12.17}$$

Taking into account the identity

$$f(x)\partial_{h_j}^- g(x) = \partial_{h_j}^-(f(x)g(x)) - (\partial_{h_j}^- f(x))g(x-he_j)$$

and summing by parts, we obtain

$$\begin{aligned}
&-\sum_{j=1}^d\int_0^\infty\int_G\int_0^t \widehat{r^{-1}}[\psi(\tau,x)](\partial_{h_j}^-\phi^j(\tau,x))\overline{v(t,x)}\, d\tau dx dt \\
&= -\sum_{j=1}^d\int_0^\infty\int_G\int_0^t \Big\{\partial_{h_j}^-(\widehat{r^{-1}}[\psi(\tau,x)](\phi^j(\tau,x)))\overline{v(t,x)} \\
&\qquad -(\partial_{h_j}^-\widehat{r^{-1}}[\psi(\tau,x)])(\phi^j(x-he_j))\overline{v(t,x)}\Big\}\, d\tau dx dt \\
&= -\sum_{j=1}^d\int_0^\infty\int_G\int_0^t \Big\{\widehat{r^{-1}}[\psi(\tau,x)](\phi^j(\tau,x))\overline{\partial_{h_j}^+ v(t,x)} \\
&\qquad +(\partial_{h_j}^-\widehat{r^{-1}}[\psi(\tau,x)])(\phi^j(x-he_j))\overline{v(t,x)}\, d\tau dx dt\,.
\end{aligned} \tag{12.18}$$

Note that the term with the integral over $\partial G$ is equal to zero because $\psi(\tau,x) \in \widehat{H}^2_{A,h}(G)$ and by virtue of Lemma 2.3. The relations (12.17) and (12.18) imply (12.16). □

Now we have to pass to the limit as $h \to 0$ in the integral

$$\int \eta(\gamma_0\psi)e^{i[L_h(\psi)v]}\nu_h(d\psi) = \int \eta(\gamma_0\psi)e^{i(f_1(\psi)+f_{2,h}(\psi)+f_{3,h}(\psi))}\nu_h(d\psi)\,. \tag{12.19}$$

To do this, we first have to study $f_{2,h}(\psi)$ and $f_{3,h}(\psi)$.

## 12.3 Investigation of $f_{2,h}(\psi)$

For $f_{2,h}(\psi)$ we prove the following result.

**Lemma 12.3.** *The following relation holds*:

$$\sum_{kh,jk\in G_h} \mathcal{X}_{Q_j}(x)|\Theta_{jk}|^2\mu_k = \sum_r \mathcal{K}_{rr}\mathcal{X}_{Q_r}(x) \to \mathcal{K}(x,x) \quad \text{as } h\to 0 \text{ a.e. } x\in G\,, \tag{12.20}$$

*where* $\mathcal{K}(x,y) = 2(\mathcal{K}_{11}(x,y) - i\mathcal{K}_{12}(x,y))$ *is the correlation function* (3.14) *of the Wiener process* $W(t,x)$ *and* $\mathcal{K}(x,x) = 2\mathcal{K}_{11}(x,x)$.

*Proof.* Recall that the matrix $\Theta_{\ell j}$ from (4.19) is unitary, i.e.,

$$\sum_k \Theta_{mk}\overline{\Theta}_{ik} = \delta_{mi} \qquad \text{and} \qquad \sum_k \Theta_{km}\overline{\Theta}_{ki} = \delta_{mi}\,. \tag{12.21}$$

We can rewrite (4.19) as follows:

$$\sum_{lr} \overline{\Theta}_{lj}\mathcal{K}_{lr}\Theta_{rk} = \delta_{jk}\mu_k\,. \tag{12.22}$$

Multiplying both parts of (12.22) by $\Theta_{mj}$, summing over $j$, and using (12.21), we obtain

$$\sum_r \mathcal{K}_{mr}\Theta_{rk} = \Theta_{mk}\mu_k\,. \tag{12.23}$$

Multiplying both sides of (12.23) by $\overline{\Theta}_{jk}$, summing over $k$, and using (12.21), we obtain

$$\mathcal{K}_{mj} = \sum_k \Theta_{mk}\overline{\Theta}_{jk}\mu_k\,. \tag{12.24}$$

Multiplying both sides of (12.24) by $\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y)$ and summing on $m,j$ such that $mh\in G_h$ and $jh\in G_h$, we obtain

$$\sum_{m,j}\mathcal{K}_{mj}\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y) = \sum_k \mu_k \sum_{m,j}\Theta_{mk}\overline{\Theta_{jk}}\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y)\,. \tag{12.25}$$

Setting $y=x$ in (12.25) and using (4.17), we obtain

$$\sum_k \mu_k \sum_m |\Theta_{mk}|^2\mathcal{X}_{Q_m}(x) = \sum_m \mathcal{K}_{mm}\mathcal{X}_{Q_m}(x) = \sum_m \mathcal{X}_{Q_m}(x)V^{-2}(Q_m)\int_{Q_m}\int_{Q_m}\mathcal{K}(x,y)\,dxdy\,. \tag{12.26}$$

Clearly, the right-hand side of (12.26) tends to $2\mathcal{K}_{11}(x,x)$ for almost all $x \in G$ as $h \to 0$. □

## 12.4 Subspaces of piecewise linear functions

The investigation of $f_{3,h}(\psi)$ is more difficult. First, we introduce the space of piecewise linear functions on $G$. For $kh \in G_h$ we consider the piecewise linear function

$$\varepsilon_k(x) = \begin{cases} 1, & x = kh, \\ 0, & x \notin \text{cube with tops } (k \pm e_j)h, j = 1, \dots, d, \\ \text{piecewise linear} & \text{otherwise.} \end{cases} \tag{12.27}$$

We define $PL_h(G)$ as the linear space of functions generated by the basis $\{\varepsilon_k(x), kh \in G_h\}$ and restricted to $G$. If this space is supplied with the norm of $L^2(G)$, we use the notation $PL_h(G)$ as well. If $PL_h(G)$ is supplied with the norm

$$\|u\|^2_{PL^1_h} = \|\nabla^+_h u\|^2_{L^2(G)} + \|u\|^2_{L^2(G)} \,,$$

we denote this space as $PL^1_h(G)$. If it is supplied with the norm

$$\|u\|^2_{PL^{2,h}} = \|\Delta_h u\|^2_{L^2(G)} + \|\nabla^+_h u\|^2_{L^2(G)} + \|u\|^2_{L^2(G)} \,,$$

then we denote this space as $PL^2_h(G)$. (For the calculation of $\nabla^+_h u$ and $\Delta_h u$ in these norms the functions $\varepsilon_k(x)$ with $kh \in \partial G^+_h$ and with coefficients from (2.23) should also be used.)

**Theorem 12.4.** *There exist constants $C_1$ and $C_2$, independent of $h$, such that for every $u \in PL^1_h(G)$*

$$C_1\|\partial_j u\|^2_{L^2(G)} \leqslant \|\partial^+_{j,h} u\|^2_{L^2(G)} \leqslant C_2\|\partial_j u\|^2_{L^2(G)}, \quad j = 1, \dots, d\,. \tag{12.28}$$

*Proof.* The estimates are established with the help of direct calculations. □

Note that the second estimate in (12.28) holds for each $u \in H^1(G)$, where, in the definition of $\partial^+_{j,h} u$, a certain extension operator $E_\delta : H^1(G) \to H^1(G(\delta)$ is used, where $G(\delta)$ is a neighborhood of $G$ with $\text{dist}(\partial G, \partial G(\delta)) = \delta$ with $\delta > 0$ is fixed.

**Theorem 12.5.** *There exists a topological isomorphism*

$$R_h : \widehat{L}^{2,h}(G) \to PL_h(G)\,. \tag{12.29}$$

*Moreover, the following estimates for the operator $R_h$ hold*:

$$\|R_h u\|_{PL^1_h(G)} \leqslant C_1 \|u\|_{\widehat{H}^1_{A,h}(G)} \leqslant C_2 \|R_h u\|_{PL^1_h(G)} \tag{12.30}$$

$$\|R_h u\|_{PL^2_h(G)} \leqslant C_1 \|u\|_{\widehat{H}^2_{A,h}(G)} \leqslant C_2 \|R_h u\|_{PL^2_h(G)} . \tag{12.31}$$

*Proof.* The isomorphism $R_h$ is established as follows. For each $u(x) \in \widehat{L}^{2,h}(G)$ we take

$$Ru(kh) = u(kh) \quad \forall\, kh \in G_h \cup \partial G_h^+ . \tag{12.32}$$

(For calculating $u(kh)$ for $kh \in \partial G_h^+$ we use the boundary conditions (2.23).) Since in both the spaces $\widehat{L}^{2,h}(G)$ and $PL_h(G)$ the values of the points $kh \in G_h \cup \partial G_h^+$ define the function uniquely for each $x \in G$, (12.32) establishes the isomorphism. The estimates (12.30) and (12.31) are proved by direct calculations. □

## *12.5 The measures $\widehat{\nu}_{h_n}$ and their weak compactness*

We need the following analogue of the compactness lemma given in Lemma 8.3.

**Lemma 12.6.** *For each $R > 0$ the set*

$$\Theta_R = \bigcup_{n=1}^{\infty} B_R(PL^2_{h_n}(G)) \cup B_R(H^2_A(G)) \tag{12.33}$$

*is compact in $H^1(G)$ if $B_R(H) = \{x \in H : \|x\|_H \leqslant R\}$ for each Hilbert space $H$.*

*Proof.* Similarly to Lemma 8.3, it suffices to choose from the sequence $u_h \in B_R(PL^2_{h_n})$ a subsequence convergent in $H^1(G)$. Clearly, we can choose a subsequence $u_m \to \widehat{u}$ weakly in $H^1(G)$ because, by virtue of (12.30) and (12.31), $u_n \in B_R(PL^2_{h_n}) \subset B_R(H^1(G))$. The following bound holds:

$$\int_G \left|\partial^-_{jh} \partial_{\ell h} \psi(x)\right|^2 dx \leqslant C \int_G \left|\partial^-_{j,h} \partial^+_{\ell h} \psi(x)\right|^2 dx \leqslant C_1 \|\psi\|_{PL^2_h(G)} , \tag{12.34}$$

where $C$ and $C_1$ do not depend on $h$. Indeed, the first inequality follows clearly from (12.28) and the second is a corollary of the discrete analogue of the elliptic theory. Recall that by the definition of $\|\psi\|_{PL^2_h(G)}$, the boundary condition for $\psi$ is fixed by (2.23). Since the right-hand side of (12.34) with $\psi = u_n$ is bounded by $C_1 R$, (12.34) implies that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $h < \delta$

$$\int |(\nabla u_m(x - e_j h) - \nabla u_m(x))^2 \, dx < \varepsilon.$$

By this inequality we can choose a subsequence $\{u_k\} \subset \{u_m\}$ strongly converging in $H^1(G)$. □

Using Lemma 12.6 analogously to Theorem 8.7, we can prove the following theorem.

**Theorem 12.7.** *For each $R > 0$ the set*

$$\widehat{\Theta}_R = \bigcup_{n=1}^{\infty} B_R(W_{h,T}) \cup B_R(W_T) \tag{12.35}$$

*is compact in $L^2(0,T;H^1(G)) \cap L^4(0,T : L^4(G)) \cap C(0,T;L^1(G))$, where*

$$\begin{aligned} W_{h,T} &= L^2(0,T;PL^2_{h_n}(G)) \cap C^L(0,T;L^1(\Omega)), \\ W_T &= L^2(0,T;H^1_\Delta(G)) \cap C^L(0,T;L^1(\Omega)) \cap L^6(0,T : L^6(G)) \end{aligned} \tag{12.36}$$

*and where $H^1_\Delta(G)$ is the space defined in* (11.16)*.*

Clearly, the isomorphism (12.29) generates the isomorphism

$$R_h : L^2(0,T;\widehat{H}^1_{A,h}(G)) \to L^2(0,T;PL^1_h(G))\,. \tag{12.37}$$

Using (12.37) and the weak solution $\nu_{h_n}(d\psi)$ defined in (9.11), we can define the following measure $\widehat{\nu}_{h,T}$ on $L^2(0,T;PL^1_h(G))$:

$$\widehat{\nu}_{hT}(B) = \nu_{hT}(R_h^{-1}B) \quad \forall\, B \in \mathcal{B}(L^2(0,T;PL^1_h(G)))\,. \tag{12.38}$$

The definition (12.38), the estimates (10.4) and (10.6) for $\nu_{hT}$, and the inequalities (11.14) and (12.31) imply the following inequality for the measures $\widehat{\nu}_{h_nT}$:

$$\int \Big( \int_0^T \|\psi(t,\cdot)\|^2_{PL^2_h(G)} dt + \|\psi\|_{C^L(0,T;L^1(G))} \Big) \widehat{\nu}_{h_nT} \leqslant C_T \tag{12.39}$$

with $C_T$ independent of $h$.

Using this estimate, the compactness result in Theorem 12.7, and the Prokhorov theorem (see Theorem 10.2), by following the proof of Lemma 10.3, we obtain the following result.

**Theorem 12.8.** *The measures $\widehat{\nu}_{h_nT}(\omega)$ are weakly compact on $L^2(0,T;H^1(G))$. Moreover,*

$$\widehat{\nu}_{h_k,T} \to \nu_T \quad \text{as } k \to \infty \text{ weakly on } L^2(0,T;H^1(G))\,, \tag{12.40}$$

*where $h_k$ is a subsequence of the sequence $h_j$ in* (10.9), $\nu_T = \Gamma *_T \nu$, *where* $\nu_T$ *is the measure* (12.40), $\nu$ *is the measure* (10.9), *and* $\Gamma_T$ *is the operator* (10.1).

*Proof.* It was already explained that $\widehat{\nu}_{h_k,T} \to \widehat{\nu}_T$ weakly on $L^2(0,T;H^1(G))$, where $\widehat{\nu}_T$ is a certain measure. To prove $\Gamma *_T \nu = \widehat{\nu}_T$, we have to take into account the fact that

$$R_h \widehat{P}_h u \to u \quad \text{as } h \to 0 \quad \forall\, u \in L^2(G)\,. \tag{12.41}$$

Indeed, by virtue of (12.38),

$$\int f(u)\widehat{\nu}_{h,T}(du) = \int f(R_h \widehat{P}_h v)\nu_{h,T}(du)$$

if $f(u)$ is continuous on $L^2\big((0,T)\times G\big)$. Passing to the limit as $h \to 0$, with the help of (12.41), we obtain $\widehat{\nu}_T = \nu_T = \Gamma *_T \nu$. □

## *12.6 The final steps for passage to the limit*

Now we are in a position to pass to the limit in (12.19). Let $N_h = R_h^{-1}$ be the operator inverse to (12.29). The equality (12.38) can be rewritten as

$$\nu_{hT}(B) = \widehat{\nu}_{h,T}(N_h^{-1}B) \quad \forall\, B \in \mathcal{B}(L^2(0,T;L^2(G)) \tag{12.42}$$

and using this, we can rewrite (12.19) in the form

$$\int \eta(\gamma_0\psi)e^{i[L_h(\psi),v]}\nu_h(d\psi) = \int \eta(\gamma_0 N_h u)e^{i[L_h(N_h u),v]}\widehat{\nu}_h(du)\,. \tag{12.43}$$

The most difficult term for passing to the limit in (12.19) as $h \to 0$ is the term $f_{3,h}(N_h u)$ from (12.16). In that integral, $u(\tau,x) \in L^2(0,T;PL_h^1(G))$. But as follows from the lemma formulated below, the operator $N_h$ can be extended from $PL_h^1(G)$ to $H^1(G)$.

**Lemma 12.9.** *The operator $N_h$ can be extended from $PL_h^1(G)$ to $H^1(G)$. Moreover, for each $u \in H^1(G)$*

$$\|\nabla_h^+ u - \nabla u\|_{L^2(G)} \to 0 \quad \textit{as } h \to 0\,. \tag{12.44}$$

*Proof.* In addition to the basis $\{\varepsilon_k(x), kh \in G_h \cup \partial G_h^+\}$, we introduce in $PL_h$ an associated basis $\{\varepsilon_k^*(x), kh \in G_h \cup \partial G_h^+\}$ that is defined by the condition

$$\int\limits_{G(\delta)} \varepsilon_j(x)\overline{\varepsilon_k^*(x)}\,dx = \delta_{kj}\,, \tag{12.45}$$

where $\delta_{kj}$ is the Kronecker symbol and $G(\delta) = \{x \in \mathbb{R}^d : \rho(x,G) = \inf_{y\in G}|x-y| < \delta\}$ is a neighborhood of $G$ containing the set $\bigcup_{kh\in G_h\cup\partial G_h^+} Q_k$ with $Q_k$ defined in (4.1)-(4.5) and (8.6). To construct $\{\varepsilon_k^*(x)\}$, we look for these functions in the form

$$\varepsilon_k^*(x) = \sum_{jh\in G_h\cup\partial G_h^+} \alpha_{kj}\varepsilon_j(x)\,, \tag{12.46}$$

where $\alpha_{k_j}$ is the solution of the system of linear algebraic equations obtained after substitution (12.46) into (12.45). By the definition of the operator $N_h$,

$$N_h f(x) = \sum_{jh\in G_h\cup\partial G_h^+} f_j\mathcal{X}_{Q_j}(x), \quad \text{where} \quad f(x) = \sum_{kh\in G_h\cup\partial G_h^+} f_k\varepsilon_k(x) \in PL_h$$

and $\mathcal{X}_{Q_j}(x)$ is the characteristic function of the set $Q_j$. The extension of this operator on $H^1(G(\delta))$ is defined as follows:

$$N_h f(x) = \sum_{kh\in G_h\cup\partial G_h^+} \mathcal{X}_{Q_j}(x)\int_{G(\delta)} f(x)\overline{\varepsilon_j^*(x)}\,dx\,. \tag{12.47}$$

The relation (12.44) is verified by direction calculations. □

Using Lemma 12.9, it is easy to prove the following result.

**Lemma 12.10.** (a) *For each sufficiently small $h$ the functional $f_{3,h}(N_h u)$ defined in* (12.16) *is continuous in $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G))$.*

(b) *For each $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G))$*

$$f_{3,h}(N_h u) \underset{h\to 0}{\longrightarrow} \int_0^\infty\int_G\int_0^t \Big\{\widehat{r^{-1}}[u(\tau,x)]\big((\nabla - iA(x))u(\tau,x)\big)\overline{\nabla v(t,x)}$$

$$-\widehat{r^{-1}}[u(\tau,x)]\big((i\nabla + A(x))u(\tau,x)\big)\overline{A(x)v(t,x)} \tag{12.48}$$

$$+\sum_{j=1}^d\big(\partial_j\widehat{r^{-1}}[u(\tau,x)]\big)\big((\nabla - iA(x))u(\tau,x)\big)\overline{v(t,x)}\Big\}\,d\tau dx dt\,.$$

Lemmas 12.3 and 12.10 imply the following assertion.

**Lemma 12.11.** (a) *For each sufficiently small $h$ the functional $[L_h(N_h u),v]$ is continuous in $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G)) \cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$.*

(b) *For each $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G)) \cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$*

$$[L_h(N_h u),v] \to [L_w(u),v] \qquad \text{as } h\to 0\,, \tag{12.49}$$

*where*

$$
\begin{aligned}
[L_w(u),v] = \int_0^\infty \int_G \Big\{ & \big(S(u(t,x) - S(\gamma_0 u(\cdot,x))\\
& + \int_0^t \widehat{r^{-1}}[u(\tau,x)](|u|^2 u(\tau,x) - u(\tau,x))\, d\tau \Big\} \overline{v}(t,x)\\
& + \int_0^t \frac{1}{2} r'[u(\tau,x)]\, d\tau K(x,x)\overline{v(t,x)}\\
& + \int_0^t \Big\{ \widehat{r^{-1}}[u(\tau,x)]\big((\nabla - iA(x))u(\tau,x)\big)\overline{\nabla v(t,x)}\\
& + \widehat{r^{-1}}[u(\tau,x)]\big((i\nabla + A(x))u(\tau,x)\big)\overline{A(x)v(t,x)}\\
& + \sum_{j=1}^d \big(\partial_j \widehat{r^{-1}}[u(\tau,x)]\big)\big((\nabla - iA(x))u(\tau,x)\big)\overline{v(t,x)}\Big\} d\tau \Big\} dxdt
\end{aligned}
\tag{12.50}
$$

*and where the index $w$ in $[L_w(u),v]$ means that* (12.50) *is the weak form of the operator $L$.*

Now we are in a position to prove the main lemma.

**Lemma 12.12.** *The following relation holds*:

$$
\int \eta(\gamma_0\psi) e^{i[L_h(\psi),v]} \nu_h(d\psi) \underset{h\to 0}{\longrightarrow} \int \eta(\gamma_0\psi) e^{i[L_w(\psi),v[} \nu(d\psi)\,, \tag{12.51}
$$

*where $\nu(d\psi)$ is the measure from* (10.9) *and $[L_w(\psi,v]$ is defined in* (12.50).

*Proof.* By virtue of (12.42), it is sufficient to prove

$$
\int \eta(\gamma_0 N_h u) e^{i[L_h(N_h u),v]} \widehat{\nu}_h(du) \underset{h\to 0}{\longrightarrow} \int \eta(\gamma_0 u) e^{i[L_w(u),v]} \nu(du)\,. \tag{12.52}
$$

Theorem 12.8 and the continuity on $L^{2,\mathrm{loc}}(0,\infty;H^1(G)) \cap C(0,\infty;L^1(G)) \cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$ of the functional $u \to e^{i[L_w(u),v]}\gamma_0(u)$ imply

$$
\int \eta(\gamma_0 u) e^{i[L_w(u),v]} \widehat{\nu}_h(du) \to \int \eta(\gamma_0, u) e^{iL_w(u),v]} \nu(du), \quad h \to 0\,. \tag{12.53}
$$

By virtue of Theorem 12.7, for each $R$, the set $\widehat{\Theta}_R$ defined in (12.35) is compact in $L^2(0,T;H^1(G)) \cap L^4((0,T)\times G) \cap C(0,T;L^1(G))$, where $T$ is chosen in such a way that $v(t,x) \equiv 0$ for $t > T$. Thus, by Lemma 12.11, for each $R > 0$,

$$
\gamma_0(N_h u) e^{i[L_h(N_h u),v]} \to \gamma_0(u) e^{i[L_w(u),v]} \quad \text{as } h \to 0 \tag{12.54}
$$

uniformly over $u \in \widehat{\Theta}_R$. In addition, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int\limits_{Q_R} \left|\gamma_0(N_h u)e^{i[L_h(N_h u),v]}\right|\widehat{\nu}_h(du) < \varepsilon \quad \forall\, h\,, \tag{12.55}$$

where $Q_k = L^2(0,T;H^1(G))\backslash\widehat{\Theta}_R$. The relations (12.53)–(12.55) imply (12.52). □

Thus, we obtain from (12.9)–(12.11) and (12.51) the equality

$$\int \eta(\gamma_0\psi)e^{i[L_w(\psi),v]}\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0)\int e^{i[W,v]}\Lambda(dW) \tag{12.56}$$

for each $v(t,x) \in L^2(0,\infty;H^1(G))$, $v(t,x) \equiv 0$ for $t > t_v$. Now we are in a position to prove (12.7).

## *12.7 Proof of the equality* (12.7)

By virtue of Theorem 11.8, the statistical solution $\nu(d\psi)$ (more precisely, its restriction $\nu_T(d\psi)$ on the time interval $(0,T)$) is supported on the space $W_T$ defined in (12.36).

**Theorem 12.13.** *The weak statistical solution $\nu(d\psi)$ satisfies Equation* (12.7) *for each $\eta \in C_b(L^2(G))$ and $\phi \in C_b(0,\infty;L^2(G))$.*

*Proof.* The main step of the proof is to show that, besides (12.56), the weak statistical solution $\nu(d\psi)$ satisfies the equality

$$\int \eta(\gamma_0\psi)e^{i[L(\psi),v]}\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0)\int e^{i[w,v]}\Lambda(dW) \tag{12.57}$$

for each $v(t,x) \in L^2(0,\infty;H^1(G))$ with $v(t,x) = 0$ for $t > t_v$, where $L(\psi)$ is the strong form of the operator $L$ defined in (12.1). Recall that $H_0^1(G) = \{u(x) \in H^1(G) : u|_{\partial G} = 0\}$; we must prove that

$$[L_w(\psi),v] = [L(\psi),v] \quad \forall\, v \in L^2(0,\infty;H_0^1(G)), v = 0 \text{ for } t > t_v. \tag{12.58}$$

By virtue of definitions (12.1) and (12.50) of $L(\psi)$ and $[L_w(\psi),v]$, to prove (12.58) we have to establish the equality

$$\int\limits_0^\infty\int\limits_G\int\limits_0^t \widehat{r^{-1}}[\psi(\tau,x)]\big((i+\nabla A(x))^2\psi(\tau,x)\;d\tau\;\overline{v(t,x)}\;dxdt$$

$$
\begin{aligned}
&= \int_0^\infty \int_G \int_0^t \Big\{ \widehat{r^{-1}}[\psi(\tau,x)]\big((\nabla - iA(x))\psi(\tau,x)\overline{\nabla v(t,x)} \\
&\quad + \widehat{r^{-1}}[\psi(\tau,x)]\big(i\nabla + A(x))\psi(\tau,x)\big)\overline{A(x)v(t,x)} \qquad\qquad (12.59)\\
&\quad + \sum_{j=1}^d \Big(\partial_j \widehat{r^{-1}}[\psi(\tau,x)]\big((\partial_j - iA(x))\psi(\tau,x)\big)\overline{v(t,x)}\Big\}\, d\tau dx dt\,.
\end{aligned}
$$

To prove this equality, one has to integrate by parts in the first term on the right-hand side and take into account that $v|_{\partial G} = 0$. This integration by parts is well justified because $\nu(d\psi) = \nu_{t_v}(d\psi)$ is supported on $W_{t_v}$ and therefore, in (12.58), $\psi \in W_{t_v}$.

Consequently, (12.56) with $v \in L^2(0,\infty; H_0^1(G))$ and (12.58) imply (12.57). Since both parts of equality (12.57) are continuous functionals with respect to $v \in L^2((0,T) \times G)$ with $v = 0$ for $t > T$ for arbitrary $T > 0$, (12.57) can be extended by continuity of $v \in L^2((0,T) \times G)$ ( $v = 0$ for $t > T$) for each $T > 0$. Now (12.7) follows from (12.57) for each cylindrical $\eta$ and $\phi$ and, after that, for arbitrary $\eta \in C_b(L^2(G))$ and $\phi \in C_b(0,\infty; L^2(G))$. □

# 13 Certain Properties of the Weak Statistical Solution $\boldsymbol{\nu}$

In this section, we show that the statistical solution $\nu(d\psi)$ is supported on solutions $\psi$ of Equation (12.1) and these solutions $\psi$ satisfy the boundary condition (2.2) on $\partial G$.

## *13.1 Boundary conditions*

The following easy assertion is true.

**Lemma 13.1.** *Let $H^1_\Delta(G)$ and $H^2_A(G)$ denote the spaces defined in* (11.16) *and* (2.5) *respectively. Then*

$$
H^2_A(G) = \{\psi \in H^1_\Delta(G) \,:\, (i\nabla + A)\psi \cdot n|_{\partial G} = 0\} \equiv \widetilde{H}\,, \qquad (13.1)
$$

*where $n$ is the unit outer normal to $\partial G$ and the last identity is the definition of $\widetilde{H}$.*

*Proof.* It is enough to prove the inclusion $\widetilde{H} \subset H^2_A(G)$ because the inverse inclusion is evident. If $\psi \in \widetilde{H}$, then

$$\Delta\psi = f \in L^2(G), \quad (i\nabla + A)\psi \cdot n|_{\partial G} = 0\,. \tag{13.2}$$

This boundary value problem is elliptic because its boundary condition satisfies the Lopatinsky condition. That is why the inequality

$$\|\psi\|_{H^2(G)} \leqslant C\|f\|_{L^2(G)} = C\|\Delta\psi\|_{L^2(G)}$$

holds, where $C$ does not depend on $\psi$. This inequality implies $\widetilde{H} \subset H^2_A(G)$. □

Recall that the space $\mathcal{U}_T$ is defined in (12.2).

**Theorem 13.2.** *For each $T > 0$ the restriction $\nu_T(d\psi)$ of the statistical solution $\nu(d\psi)$ on the time interval $(0, T)$ is supported on the space $\mathcal{U}_T$.*

*Proof.* Since $\nu_T(d\psi)$ is supported on the space $W_T$ defined in (12.36), we have to prove, by virtue of Lemma 13.1, that there exists a $\nu_T(d\psi)$-measurable set $\mathcal{F} \subset W_T$ such that $\nu_T(\mathcal{F}) = 1$ and $(i\nabla+A)\psi\cdot n|_{\partial G} = 0$ for each $\psi \in \mathcal{F}$. Taking $\eta \equiv 1$ in (12.56), we differentiate this equality twice on $v \in L^2(0,\infty; H^1(G))$ such that $v(t,x) \equiv 0$ for $t \geqslant T$. As a result, we obtain

$$\int [L_w(\psi)u]^2 e^{i[L_w(\psi),v]}\nu_T(d\psi) = \int [W,u]^2 e^{i[w,v]}\Lambda_T(dW)\,, \tag{13.3}$$

where $u \in L^2(0,T; H^1(G))$ is arbitrary. We take $v \equiv 0$ in (13.3) and then integrate by parts on the left-hand side of this equality as we did in (12.59). This integration by parts is well-justified because the inclusion $u \in L^2(0,T; H^1(G))$ implies that $u|_{\partial G} \in L^2(0,T; H^{1/2}(\partial G))$ and, as is well-known (see [17, 31]), the inclusion $\psi \in L^2(0,T; H^1_\Delta(G))$ implies $(i\nabla + A)\psi \cdot n|_{\partial G} \in L^2(0, T : H^{-1/2}(\partial G))$.

Since $u|_{\partial G} \neq 0$, in contrast to (12.58), after integration by parts we obtain

$$\begin{aligned} \int \Big( \int_0^T \int_{\partial G} \int_0^t \widehat{r^{-1}}[\psi(\tau,x)](\nabla - iA(x))\psi(\tau,x)) \cdot n \,\overline{u(t,x)}\, d\tau dx dt \\ +[L(\psi),u]\Big)^2 \nu_T(du) = \int [W,u]^2\Lambda_T(dW)\,. \end{aligned} \tag{13.4}$$

Instead of $u(t,x)$in (13.4), we now take the sequence $u_n(t,x)$ that satisfies the properties:

a. $u_n(t,x) \to 0$ in $L^2((0,T)\times G)$;
b. for each $n$, $u_n(t,x)|_{\partial G} = \partial_t v(t,x)$, where $v(t,x) \in H^1_0(0,T; H^{1/2}(\partial G)$ is fixed.

Passing to the limit in (13.4) as $n \to \infty$ and after that integrating by parts on the left-hand side of the resulting equality, we obtain

$$\int \Big( \int_0^T \int_{\partial G} r^{-1}[\psi(t,x)]\big((\nabla - iA(x))\psi(t,x)\cdot n\big)\overline{v(t,x)}\,dxdt\Big)^2 \nu_T(du) = 0\,. \tag{13.5}$$

Now we choose a countable dense set $\{v_n\}$ in $L^2(0,T;H^{1/2}(G))$ and, for each $n$, put $v_n$ in (13.5). As a result, for each $n$ we obtain the measurable set $\mathcal{F}_n \subset W_T$ such that

$$\nu_T(\mathcal{F}_n) = 1\,,$$

$$\int_0^T \int_{\partial G} r^{-1}[\psi(t,x)]\big((\nabla - iA)\psi(t,x)\cdot n\big)\overline{v_n(t,x)}\,dxdt = 0 \quad \forall\, \psi \in \mathcal{F}_n\,. \tag{13.6}$$

We take $\mathcal{F} = \bigcap\limits_n \mathcal{F}_n$. Clearly, $\nu_T(\mathcal{F}) = 1$ and

$$\widehat{r^{-1}}[\psi(t,x)]\big((\nabla - iA(x))\psi(t,x)\cdot n\big)|_{(0,T)\times\partial G} = 0 \quad \forall\, \psi \in \mathcal{F}\,. \tag{13.7}$$

Since $r^{-1}(\mathrm{Re}\,\psi(t,x)) > 0$ and $r^{-1}(\mathrm{Im}\psi(t,x)) > 0$ for all $(t,x) \in (0,T)\times\overline{G}$, (13.7) implies

$$\nu_T(\mathcal{F}) = 1, \quad (i\nabla + A)\psi(t,x)\cdot n|_{(0,T)\times\partial G} = 0 \quad \forall\, \psi \in \mathcal{F}\,.$$

These equalities complete the proof of the theorem. □

## 13.2 Solvability for almost all data

Recall that the initial measure $\mu$ is supported on the space $H^1(G)$ and the Wiener measure $\Lambda$ is supported on the set $\widehat{W}$ defined in (9.17).

**Theorem 13.3.** (a) *For $\mu\times\Lambda$-almost all data $(\psi_0, W)$ there exists a solution $\psi \in \mathcal{U}$ of the problem* (12.1).

(b) *The weak statistical solution $\nu$ is supported on solutions of the problem* (12.1) *and* (2.2).

*Proof.* Since $\mathcal{U}$ defined in (12.3) is a separable Frechet space, by the Riesz theorem (see [19]), for any $N > 0$ there exists a compact set $K_N \subset \mathcal{U}$ such that

$$\nu(K_N) \geqslant 1 - \frac{1}{N}\,. \tag{13.8}$$

The continuity of the operator (12.5) implies that $\mathfrak{A}K_N$ is compact in $L^1(G)\times Z$ and therefore $\mathfrak{A}K_N \in \mathcal{B}(L^1(G)\times Z)$. We set

$$F_N = \mathfrak{A}K_N \cap \{H^1(G) \times \widehat{W}\}, \quad F = \bigcup_{N=1}^{\infty} F_N\,. \tag{13.9}$$

Since $H^1(G) \in \mathcal{B}(L^1(G))$ (see [44, Theorem 2.1]) and the set $\widehat{W}$ is $\Lambda$-measurable, each set from (13.9) is $\mu \times \Lambda$-measurable. By virtue of (12.6),

$$\nu(\mathfrak{A}^{-1}(H^1(G) \times \widehat{W})) = \mu(H^1(G)) \cdot \Lambda(\widehat{W}) = 1\,. \tag{13.10}$$

Thus, taking into account (13.8)-(13.10), we obtain

$$\begin{aligned} \mu \times \Lambda(F) = \nu(\mathfrak{A}^{-1}F) &\geqslant \nu(\mathfrak{A}^{-1}(H^1(G) \times \widehat{W}) \cap \bigcup_{N=1}^{\infty} K_N) \\ &= \nu(\bigcup_{N=1}^{\infty} K_N) \geqslant \lim_{N\to\infty} \nu(K_N) = 1\,. \end{aligned} \tag{13.11}$$

Directly from the definition $\mathfrak{A}^{-1}F = \{\psi \in \mathcal{U} \,:\, \mathfrak{A}\psi \in F\}$, we obtain

$$F \in \mathfrak{A}\mathcal{U}\,. \tag{13.12}$$

The relations (13.11) and (13.12) prove statement (a) of the theorem. We set

$$K = \big(\bigcup_{N=1}^{\infty} K_N\big) \cap \mathfrak{A}^{-1}(H^1(G) \times \widehat{W})\,. \tag{13.13}$$

The relations (13.8), (13.10), and (13.13) imply $\nu(K) = 1$, and the relations (13.9) and (13.12) imply that $\mathfrak{A}K = F$. The last two relations prove statement (b) of the theorem. □

## 14 Uniqueness of the Weak Statistical Solution

The main step in proving the uniqueness of a weak statistical solution for the stochastic Ginzburg–Landau problem is a proof of uniqueness for (12.1) with fixed (non-stochastic) data $(\psi_0(x), W(t,x))$.

### *14.1 Reduction of uniqueness for statistical solution $\nu$ to uniqueness of the solution for* (12.1)

Let $F$ and $K$ be the sets (13.9) and (13.13) respectively. In Theorem 13.3, we proved that the set $F$ is $\mu \times \Lambda$-measurable, $K$ is $\nu$-measurable,

$$(\mu \times \Lambda)(F) = 1, \quad \nu(K) = 1, \quad \text{and} \quad \mathfrak{A}K = F\,, \tag{14.1}$$

where $\mathfrak{A}$ is the operator (12.5), $\nu$ is a weak statistical solution, $\mu$ is the initial measure, and $\Lambda$ is the Wiener measure.

**Lemma 14.1.** *If, for each initial datum $(\psi_0, W) \in F$, an individual solution $\psi$ of the problem* (12.1) *and* (2.2) *is unique in $K$, then the statistical solution $\nu$ of the stochastic Ginzburg–Landau problem* (3.22), (2.2), *and* (2.3) *is unique.*

*Proof.* In Theorem 13.3, we proved that each weak statistical solution $\nu$ corresponding to the given initial measure $\mu$ and the Wiener measure $\Lambda$ is supported on the set $K$ defined in (13.13). Since for each datum $(\psi_0, W) \in F$, the solution $\psi$ of (12.1) and (2.2) is unique in $K$, the full preimage

$$\mathfrak{A}^{-1}F = \{\psi \in K \,:\, \mathfrak{A}\psi \in F\} \tag{14.2}$$

consists of the unique element $\psi \in K$ for each given datum $(\psi_0, W) \in F$. Therefore, a weak statistical solution $\nu(d\psi)$ is defined uniquely by the formula

$$\nu(B) = \nu(B \cap K) = \mu(\gamma_0 B)\Lambda(LB) \quad \forall\, B \in \mathcal{B}(\mathcal{U})\,. \tag{14.3}$$

□

## *14.2 Proof of the uniqueness of the solution of* (12.1) *and* (2.2)*: the first step*

Suppose that for a given datum $(\psi_0, W) \in F$ there exist two solutions $\psi_i(t,x) \in K$, $i = 1, 2$, of the problem (12.1) and (2.2). Then

$$L(\psi_1) - L(\psi_2) = 0, \qquad (\psi_1 - \psi_2)|_{t=0} = 0\,, \tag{14.4}$$

where $L$ is the operator defined in (12.1). Denote

$$\sigma(t,x) = S[\psi_1(t,x)] - S[\psi_2(t,x)]\,. \tag{14.5}$$

Since $\psi_i \in K \subset \mathcal{U}$, $i = 1, 2$, where $\mathcal{U}$ is the space (12.3), the relations (12.1) and (14.4) imply that for each $T > 0$, $\sigma(t,x) \in H^1(0,T; L^2(G))$, i.e., $\sigma$ is differentiable in $t$. Thus, we can differentiate both parts of (14.4) with respect to $t$. Doing this, we obtain by (12.1):

$$\begin{aligned} &\partial_t\sigma(t,x) + \widehat{r^{-1}}[\psi_1]\{(i\nabla + A)^2\psi_1 - \psi_1 + |\psi_1|^2\psi_1\} \\ &\quad -\widehat{r^{-1}}[\psi_2]\{(i\nabla + A)^2\psi_2 - \psi_2 + |\psi_2|^2\psi_2)\} \\ &\quad +(r'[\psi_1] - r'[\psi_2])\mathcal{K}_{11}(x,x) = 0\,. \end{aligned} \tag{14.6}$$

Multiplying (14.6) by $\overline{\sigma(t,x)}$ and integrating over $G$, we obtain

$$\frac{1}{2}\partial_t\|\sigma(t,\cdot)\|^2_{L^2(G)} + T_1 + T_2 + T_3 + T_4 = 0\,, \tag{14.7}$$

where

$$T_1 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{(i\nabla + A)^2\psi_1\} - \widehat{r^{-1}}[\psi_2]\{(i\nabla + A)^2\psi_2\}\Big)\overline{\sigma}\; dx\,, \tag{14.8}$$

$$T_2 = -\int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\psi_1 - \widehat{r^{-1}}[\psi_2]\psi_2\Big)\overline{\sigma}\; dx\,, \tag{14.9}$$

$$T_3 = \int\limits_G \Big(r'[\psi_1] - r'[\psi_2]\Big)\{\mathcal{K}_{11}(x,x)\}\overline{\sigma}\; dx\,, \tag{14.10}$$

and

$$T_4 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{|\psi_1|^2\psi_1\} - \widehat{r^{-1}}[\psi_2]\{|\psi_2|^2\psi_2\}\Big)\overline{\sigma}\; dx\,. \tag{14.11}$$

Taking into account

$$\nabla_x S[\psi(t,x)] = \widehat{r^{-1}}[\psi(t,x)]\nabla_x\psi(t,x) \tag{14.12}$$

and performing a transformation analogous to the one in (12.59), we obtain

$$T_1 = \int\limits_G |\nabla_x\sigma(t,x)|^2\; dx + T_5 + T_6 + T_7 + T_8 + T_9\,, \tag{14.13}$$

where

$$T_5 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_2]\{iA\psi_2\} - \widehat{r^{-1}}[\psi_1]\{iA\psi_1\}\Big)\cdot\overline{\nabla\sigma}\; dx\,, \tag{14.14}$$

$$T_6 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{i\nabla\psi_1\} - \widehat{r^{-1}}[\psi_2]\{i\nabla\psi_2\}\cdot A(x)\overline{\sigma}\; dx\,, \tag{14.15}$$

$$T_7 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{A\psi_1\} - \widehat{r^{-1}}[\psi_2]\{A(x)\psi_2\}\cdot A(x)\overline{\sigma}\; dx\,, \tag{14.16}$$

$$T_8 = \int\limits_G \Big(\sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_1]\{\partial_j\psi_1\} - \sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_2]\{\partial_j\psi_2\}\Big)\overline{\sigma}\; dx\,, \tag{14.17}$$

and

$$T_9 = \int_G \Big( \sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_1]\{iA^j\psi_1\} - \sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_2]\{iA^j\psi_2\}\Big)\overline{\sigma}\, dx\,. \tag{14.18}$$

We estimate these terms in the following three subsections.

## 14.3 Estimation of the terms $T_2$ to $T_5$, $T_7$, and $T_9$

We begin with a generalization of the bound (7.25). Let $r(\lambda)$, $S(\lambda)$, and $R(\lambda)$ be the functions (3.19), (7.7), and (7.23) respectively. Since by (7.23) we have $\lambda = R(S(\lambda))$, we obtain

$$1 = R'(S(\lambda))S'(\lambda) = \frac{R'(S(\lambda))}{r(\lambda)} \Rightarrow R'(S(\lambda)) = r(\lambda)\,, \tag{14.19}$$

where we have used (7.7). Therefore, for a real-valued function $f(\lambda) \in C^1(\mathbb{R}^1)$, we obtain, by the Lagrange theorem and (14.19),

$$f(\lambda_2) - f(\lambda_1) = f(R(S_2)) - f(R(S_1)) \leqslant \sup_{\lambda\in[\lambda_1,\lambda_2]} |f'(\lambda)r(\lambda)|\,|S_2 - S_1|\,, \tag{14.20}$$

where we have used the notation $S_i = S(\lambda_i)$, $i = 1, 2$. For $f(\lambda) = \lambda/r(\lambda)$ the function $f'(\lambda)r(\lambda)$ is bounded and therefore, by (14.20), (3.20), and (3.21), the term (14.9) admits the bound

$$|T_2| \leqslant C \int_G |\sigma(t,x)|^2\, dx\,. \tag{14.21}$$

Since $A(x) \in C^2(\overline{G})$, we obtain in an analogous manner that

$$|T_7| \leqslant C \int_G |\sigma(t,x)|^2\, dx \tag{14.22}$$

and

$$|T_3| \leqslant C \int |\mathcal{K}_{11}(x,x)||\sigma(t,x)|^2\, dx\,. \tag{14.23}$$

We impose on the correlation function $\mathcal{K}_{11}(x,x)$ the following additional condition:[4]

[4] Note that when $\dim G = 2$, condition (14.24) follows from condition (3.17). Indeed, using the well-known representation $\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}$ of the trace class kernel, one can easily derive from (3.17) that $\mathcal{K}(x,x) \in W_1^1(G) \subset L^2(G)$ (the last enclosure follows from Sobolev embedding theorem).

$$\mathcal{K}_{11}(x,x) \in L^p(G) \text{ with } p > 1 \text{ if } \dim G = 2 \text{ and} \\ \text{with } p > \tfrac{3}{2} \text{ if } \dim G = 3. \tag{14.24}$$

Suppose that $d = \dim G = 2$. Using the Sobolev embedding theorem ($H^s(G) \subset L^q(G)$ for $s \geqslant d(\frac{1}{2} - \frac{1}{q})$), the interpolation inequality $\|u\|_{H^s} \leqslant C\|u\|_{L^2}^{1-s}\|u\|_{H^1}^s$ for $0 < s < 1$, and the notation $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain from (14.23) that

$$\begin{aligned} |T_3| &\leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{L^{2q}}^2 \leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{H^{1-1/q}}^2 \\ &\leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{L^2}^{2/q}\|\sigma\|_{H^1}^{2-2/q} \leqslant \varepsilon\|\sigma\|_{H^1}^2 + C_\varepsilon\|\mathcal{K}_{11}\|_{L^p}^q\|\sigma\|_{L^2}^2 . \end{aligned} \tag{14.25}$$

The proof of (14.25) in the case $d = \dim G = 3$ is absolutely the same. Doing elementary algebraic transformations and using (14.20) and the Sobolev embedding theorem ($C(\overline{G}) \subset H^2(G)$ for $d \leqslant 3$), we obtain

$$\begin{aligned} |T_4| &\leqslant \int \Big(\big|\widehat{r^{-1}}[\psi_1]\psi_1 - \widehat{r^{-1}}[\psi_2]\psi_2\big|\ |\psi_1|^2 \\ &\quad + \big|\widehat{r^{-1}}[\psi_2]\psi_2\big|(|\psi_1|^2 - |\psi_2|^2)\Big)|\sigma|\ dx \\ &\leqslant C\int \big(|\psi_1|^2|\sigma|^2 + (|\psi_1|^2 + |\psi_2|^2)|\sigma|^2\big)\ dx \\ &\leqslant C\big(1 + \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^2}^2\big)\textstyle\int |\sigma|^2\ dx \end{aligned} \tag{14.26}$$

if $d = \dim G \leqslant 3$.

After elementary transformations, we obtain by (14.20) and the Sobolev embedding theorem

$$\begin{aligned} |T_5| &\leqslant C\int\limits_G \Bigg(\left|\frac{\operatorname{Im}\psi_2}{r(\operatorname{Re}\psi_2)} - \frac{\operatorname{Im}\psi_1}{r(\operatorname{Re}\psi_1)}\right| \\ &\quad + \left|\frac{\operatorname{Re}\psi_2}{r(\operatorname{Im}\psi_2)} - \frac{\operatorname{Re}\psi_1}{r(\operatorname{Im}\psi_1)}\right|\Bigg)|A\cdot\nabla\sigma|\ dx \\ &\leqslant C\int\limits_G \Bigg(\frac{|\operatorname{Im}\psi_2 - \operatorname{Im}\psi_1|}{r(\operatorname{Re}\psi_2)} + |\operatorname{Im}\psi_1|\left|\frac{1}{r(\operatorname{Re}\psi_1)} - \frac{1}{r(\operatorname{Re}\psi_2)}\right| \\ &\quad + \frac{|\operatorname{Re}\psi_2 - \operatorname{Re}\psi_1|}{r(\operatorname{Im}\psi_1)} + |\operatorname{Re}\psi_1|\left|\frac{1}{r(\operatorname{Im}\psi_1)} - \frac{1}{r(\operatorname{Im}\psi_2)}\right|\Bigg)|\nabla\sigma|\ dx \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int\limits_G (1+|\psi_1|+|\psi_2|)(|S(\operatorname{Re}\psi_1)-S(\operatorname{Re}(\psi_2)| \\
&\qquad +|S(\operatorname{Im}\psi_1)-S(\operatorname{Im}(\psi_2)|\Big)\ |\nabla\sigma|\ dx \\
&\leqslant C(1+\|\psi_1\|_{H^2(G)}+\|\psi_2\|_{H^2(G)})\int|\sigma|\ |\nabla\sigma|\ dx \\
&\leqslant \varepsilon\|\nabla\sigma\|^2_{L^2(G)}+C_\varepsilon(1+\|\psi_1\|_{H^2(G)}+\|\psi_2\|_{H^2(G)})\int|\sigma|^2\ dx\,.
\end{aligned}
\tag{14.27}
$$

To bound $T_9$, we first do some simple transformations using (14.12) to obtain

$$
\begin{aligned}
T_9=-\int\limits_G \Big\{\Big(&\frac{r'(\operatorname{Re}\psi_1)}{r(\operatorname{Re}\psi_1)}\big(\nabla S(\operatorname{Re}\psi_1)\cdot A\big)\operatorname{Im}\psi_1 \\
&-\frac{r'(\operatorname{Re}\psi_2)}{r(\operatorname{Re}\psi_2)}\big(\nabla S(\operatorname{Re}\psi_2)\cdot A\big)\operatorname{Im}\psi_2\Big) \\
&-i\frac{r'(\operatorname{Im}\psi_1)}{r(\operatorname{Im}\psi_1)}\big(\nabla S(\operatorname{Im}\psi_1)\cdot A\big)\operatorname{Re}\psi_1 \\
&-i\frac{r'(\operatorname{Im}\psi_2)}{r(\operatorname{Im}\psi_2)}\big(\nabla S(\operatorname{Im}\psi_2)\cdot A\big)\operatorname{Re}\psi_2\Big)\Big\}\overline{\sigma}\ dx
\end{aligned}
$$

so that

$$
\begin{aligned}
T_9=-\int\limits_G \Big(\Big\{\Big(&\frac{r'(\operatorname{Re}\psi_1)}{r(\operatorname{Re}\psi_1)}\nabla\operatorname{Re}\sigma\cdot A\operatorname{Im}\psi_1+\nabla S(\operatorname{Re}\psi_2)\cdot \\
&A\Big[\Big(\frac{r'(\operatorname{Re}\psi_1)}{r(\operatorname{Re}\psi_1)}-\frac{r'(\operatorname{Re}\psi_2)}{r(\operatorname{Re}\psi_2)}\Big)\operatorname{Im}\psi_1+\frac{r'(\operatorname{Re}\psi_2)}{r(\operatorname{Re}\psi2)}(\operatorname{Im}\psi_1-\operatorname{Im}\psi_2)\Big]\Big\} \\
&-i\Big\{\frac{r'(\operatorname{Im}\psi_1)}{r(\operatorname{Im}\psi_1)}\nabla\operatorname{Im}\sigma\cdot A\operatorname{Re}\psi_1 \\
&+\nabla S(\operatorname{Im}\psi)\cdot A\Big[\Big(\frac{r'(\operatorname{Im}\psi_1)}{r(\operatorname{Im}\psi_1)}-\frac{r'(\operatorname{Im}\psi_2)}{r(\operatorname{Im}\psi_2)}\Big)\operatorname{Re}\psi_1 \\
&+\frac{r'(\operatorname{Im}\psi_2)}{r(\operatorname{Im}\psi_2)}(\operatorname{Re}\psi_1-\operatorname{Re}\psi_2)\Big]\Big\}\Big)\overline{\sigma}\ dx\,.
\end{aligned}
\tag{14.28}
$$

A simple bound of the right-hand side of (14.28) and the use of (14.20) gives

$$|T_9| \leqslant C \int_G |\nabla\sigma|\ |\psi_1|\ |\sigma| + |\nabla\psi_2|\ (1+|\psi_1|+|\psi_2|)\ |\sigma|^2\ dx\,. \tag{14.29}$$

Using the same tools as in (14.25), we have (when dim $G \leqslant 3$)

$$\begin{aligned}
&\int (1+|\psi_1|+|\psi_2|)|\nabla\psi_2|\ |\sigma|^2\ dx \\
&\leqslant C\big(1+\|\psi_1\|_{L^6}+\|\psi_2\|_{L^6}\big)\|\nabla\psi_2\|_{L^3}\|\sigma\|_{L^4}^2 \\
&\leqslant C\big(1+\|\nabla\psi_1\|_{L^2}+\|\nabla\psi_2\|_{L^2}\big)\|\nabla\psi_2\|_{H^{1/2}}\|\sigma\|_{H^{3/4}}^2 \\
&\leqslant C\big(1+\|\nabla\psi_1\|_{L^2}+\|\nabla\psi_2\|_{L^2}\big)\|\nabla\psi_2\|_{L^2}^{1/2}\|\psi_2\|_{H^2}^{1/2}\|\sigma\|_{L^2}^{1/2}\|\sigma\|_{H^1}^{3/2} \\
&\leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 \\
&\quad +C_\varepsilon\big(1+\|\nabla\psi_1\|_{L^2}+\|\nabla\psi_2\|_{L^2}\big)^4\|\nabla\psi_2\|_{L^2}^2\|\psi_2\|_{H^2}^2\|\sigma\|_{L^2}^2\,.
\end{aligned} \tag{14.30}$$

Using (14.29) and (14.30), we obtain

$$|T_9| \leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 + C_\varepsilon\big(1+\|\nabla\psi_1\|_{L^2}+\|\nabla\psi_2\|_{L^2}\big)^4\|\nabla\psi_2\|_{L^2}^2\|\psi_2\|_{H^2}^2\|\sigma\|_{L^2}^2\,. \tag{14.31}$$

## *14.4 Estimation of $T_6$ and $T_8$*

Using (14.12), we obtain

$$\begin{aligned}
T_6 = \int_G \Bigg(&\bigg(\frac{\nabla \mathrm{Im}\,\psi_2}{r(\mathrm{Re}\,\psi_2)} - \frac{\nabla \mathrm{Im}\,\psi_1}{r(\mathrm{Re}\,\psi_1)}\bigg) \\
&+ i\bigg(\frac{\nabla \mathrm{Re}\,\psi_1}{r(\mathrm{Im}\,\psi_1)} - \frac{\nabla \mathrm{Re}\,\psi_2}{r(\mathrm{Im}\,\psi_2)}\bigg)\Bigg)\cdot A\overline{\sigma}\ dx
\end{aligned}$$

so that

$$T_6 = \int_G \Bigg\{ \bigg(\frac{r(\mathrm{Im}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}\nabla S(\mathrm{Im}\,\psi_2) - \frac{r(\mathrm{Im}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}\nabla S(\mathrm{Im}\,\psi_1)\bigg)$$

$$-i\Big(\frac{r(\operatorname{Re}\psi_2)}{r(\operatorname{Im}\psi_2)}\nabla S(\operatorname{Re}\psi_2)-\frac{r(\operatorname{Re}\psi_1)}{r(\operatorname{Im}\psi_1)}\nabla S(\operatorname{Re}\psi_1)\Big)\Bigg\}\cdot A\overline{\sigma}\,dx$$

$$=\int\limits_G\Bigg\{\Big(\frac{-r(\operatorname{Im}\psi_2)}{r(\operatorname{Re}\psi_2)}\nabla\operatorname{Im}\sigma+\Big(\frac{r(\operatorname{Im}\psi_2)}{r(\operatorname{Re}\psi_2)}-\frac{r(\operatorname{Im}\psi_1)}{r(\operatorname{Re}\psi_1)}\Big)\nabla S(\operatorname{Im}\psi_1)\Big)$$

$$-i\Big(\frac{-r(\operatorname{Re}\psi_2)}{r(\operatorname{Im}\psi_2)}\nabla\operatorname{Re}\sigma \tag{14.32}$$

$$+\Big(\frac{r(\operatorname{Re}\psi_2)}{r(\operatorname{Im}\psi_2)}-\frac{r(\operatorname{Re}\psi_1)}{r(\operatorname{Im}\psi_1)}\Big)\nabla S(\operatorname{Re}\psi_1)\Big)\cdot A\overline{\sigma}\,dx\,.$$

Estimating with the help of (14.20), we obtain the bound

$$|T_6|\leqslant\int\limits_G\big\{(1+|\psi_2|)|\nabla\sigma|+|r(\operatorname{Im}\psi_2)r(\operatorname{Re}\psi_1)$$

$$-r(\operatorname{Re}\psi_2)r(\operatorname{Im}\psi_1)|\;|\nabla\psi_1|\big\}|\sigma|\;dx$$

$$\leqslant\int\limits_G\big\{(1+|\psi_2|)|\nabla\sigma|+\Big(|r(\operatorname{Im}\psi_2)-r(\operatorname{Im}\psi_1)|r(\operatorname{Re}\psi_1) \tag{14.33}$$

$$+r(\operatorname{Im}(\psi_1)|r(\operatorname{Re}\psi_1)-r(\operatorname{Re}(\psi_2)|\Big)|\nabla\psi_1|\big\}|\sigma|\;dx$$

$$\leqslant C\int\limits_G(1+|\psi_2|)|\nabla\sigma|\;|\sigma|+(1+|\psi_1|^2+|\psi_2|^2)|\nabla\psi_1|\;|\sigma|^2\;dx\,.$$

We assume now that $d=\dim G\leqslant 2$. Then

$$\int(1+|\psi_1|^2+|\psi_2|^2)|\nabla\psi_1|\;|\sigma|^2\;dx$$

$$\leqslant C\big(1+\|\psi_1\|^2_{L^{12}}+\|\psi_2\|^2_{L^{12}}\big)\|\nabla\psi_1\|_{L^3}\|\sigma\|^2_{L^4}$$

$$\leqslant C\big(1+\|\psi_1\|^2_{H^1}+\|\psi_2\|^2_{H^1}\big)\|\psi_1\|_{H^{4/3}}\|\sigma\|^2_{H^{1/2}}$$

$$\leqslant C\big(1+\|\psi_1\|^2_{H^1}+\|\psi_2\|^2_{H^1}\big)\|\psi_1\|^{2/3}_{H^1}\|\psi_1\|^{1/3}_{H^2}\|\sigma\|_{L^2}(\|\nabla\sigma\|_{L^2}+\|\sigma\|_{L^2})$$

$$\leqslant\varepsilon\|\nabla\sigma\|^2_{L^2}+C_\varepsilon\big(1+\|\psi_1\|_{H^1}+\|\psi_2\|_{H^1}\big)^{16/3}\|\psi_1\|^{2/3}_{H^2}\|\sigma\|^2_{L^2}\,. \tag{14.34}$$

Now (14.33) and (14.34) imply

$$
\begin{aligned}
|T_6| &\leqslant \varepsilon \|\nabla\sigma\|_{L^2}^2 \\
&\quad + C_\varepsilon \Big\{ \big(1 + \|\psi_1\|_{H^1} + \|\psi_2\|_{H^1}\big)^{16/3} \|\psi_2\|_{H^2})^{2/3} \Big\} \|\sigma\|_{L^2}^2 \,.
\end{aligned}
\tag{14.35}
$$

Finally we estimate the term $T_8$. By virtue of (3.20), (3.21), and (14.5) and through the use of the notation $[\nabla S]^2 = |\nabla \mathrm{Re}\, S|^2 + i|\nabla \mathrm{Im}\, S|^2$, we can write

$$
\begin{aligned}
T_8 &= \int\limits_G \Big( \widehat{r'}[\psi_1][\nabla S[\psi_1]]^2 - \widehat{r'}[\psi_2][\nabla S[\psi_2]]^2 \Big) \overline{\sigma}\, dx \\
&= \int\limits_G \Big( (r'[\widehat{\psi_1] - r'}[\psi_2][\nabla S[\psi_1]]^2 + \widehat{r'}[\psi_2]([\nabla S[\psi_1]]^2 - [\nabla S[\psi_2]]^2) \Big) \overline{\sigma}\, dx \,.
\end{aligned}
\tag{14.36}
$$

Bounding (14.36) with the help of (14.20) and (14.5), we obtain

$$
|T_8| \leqslant \int\limits_G \Big( |\sigma|^2 |\nabla\psi_1|^2 + |\nabla\sigma|\ |\sigma|(|\nabla\psi_1| + |\nabla\psi_2|) \Big)\, dx \,. \tag{14.37}
$$

Assume that $d$ =dim $G \leqslant 2$. Then

$$
\begin{aligned}
\int\limits_G |\sigma|^2 |\nabla\psi_1|^2\, dx &\leqslant \|\sigma\|_{L^4}^2 \|\nabla\psi_1\|_{L^4}^2 \leqslant \|\sigma\|_{H^{1/2}}^2 \|\nabla\psi_1\|_{H^{1/2}}^2 \\
&\leqslant \|\sigma\|_{L^2} \|\sigma\|_{H^1} \|\nabla\psi_1\|_{L^2} \|\psi_1\|_{H^2} \\
&\leqslant \varepsilon(\|\nabla\sigma\|_{L^2}^2 + \|\sigma\|_{L^2}^2) + C_\varepsilon \|\sigma\|_{L^2}^2 \|\nabla\psi_1\|_{L^2}^2 \|\psi_1\|_{H^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\int\limits_G |\nabla\sigma|\ |\sigma|(|\nabla\psi_1| + |\nabla\psi_2|)\, dx &\leqslant \|\sigma\|_{L^6} \|\nabla\sigma\|_{L^2} (\|\nabla\psi_1\|_{L^3} + \|\nabla\psi_2\|_{L^3}) \\
&\leqslant \|\sigma\|_{H^{2/3}} \|\nabla\sigma\|_{L^2} \big( \|\nabla\psi_1\|_{H^{1/3}} + \|\nabla\psi_2\|_{H^{1/3}} \big) \\
&\leqslant \Big( \|\sigma\|_{L^2}^{1/3} \|\nabla\sigma\|_{L^2}^{5/3} + \|\nabla\sigma\|_{L^2} \|\sigma\|_{L^2} \Big) \\
&\qquad \Big( \|\nabla\psi_1\|_{L^2}^{2/3} \|\psi_1\|_{H^2}^{1/3} + \|\nabla\psi_2\|_{L^2}^{2/3} \|\psi_2\|_{H^2}^{1/3} \Big) \\
&\leqslant \varepsilon \|\nabla\sigma\|_{L^2}^2 + C_\varepsilon \|\sigma\|_{L^2}^2 \Big( \|\nabla\psi_1\|_{L^2}^4 \|\psi_1\|_{H^2}^2 + \|\nabla\psi_2\|_{L^2}^4 \|\psi_2\|_{H^2}^2 + 1 \Big) \,.
\end{aligned}
$$

The last two inequalities and (14.37) imply

$$|T_8| \leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 + C_\varepsilon\|\sigma\|_{L^2}^2\left(\|\nabla\psi_1\|_{L^2}^4\|\psi_1\|_{H^2}^2 + \|\nabla\psi_2\|_{L^2}^4\|\psi_2\|_{H^2}^2 + 1\right). \tag{14.38}$$

*Remark* 14.1. We estimated all the terms except $T_6$ and $T_8$ under the assumption that $d = \dim\, G \leqslant 3$. We cannot estimate the terms $T_6$ and $T_8$ under this assumption. We are forced to assume that $d = \dim\, G \leqslant 2$ when we bound $T_6$ and $T_8$.

## *14.5 Uniqueness theorems*

We are now in a position to prove a uniqueness theorem for individual solutions of the problem (12.1) and (2.2).

**Theorem 14.2.** *Let $d = \dim G = 2$, and let the correlation function $\mathcal{K}_{11}(x,x)$ for the Wiener measure $\Lambda$ belong to $L^p(G)$ with a certain $p > 1$.*[5] *Then for each datum $(\psi_0, W) \in F$ a solution $\psi \in K$ of the problem* (12.1) *and* (2.2) *is unique.* (*Here $F$ and $K$ are the sets defined in* (13.9) *and* (13.13) *respectively.*)

*Proof.* Assume that, for a datum $(\psi_0, W) \in F$ there exist two solutions $\psi_1$ and $\psi_2$. Then for the function $\sigma$ defined in (14.5) the following estimate is derived from (14.7) and (14.13):

$$\frac{1}{2}\partial_t\|\sigma(t,\cdot)\|_{L^2}^2 + \int_G |\nabla_x\sigma(t,x)|^2\, dx \leqslant |T_2| + \cdots + |T_9|\,. \tag{14.39}$$

Using the estimates (14.21), (14.22), (14.25)–(14.27), (14.31), (14.35), and (14.38), we obtain

$$\begin{aligned}&\partial_t\|\sigma(t,\cdot)\|_{L^2}^2 + \|\nabla\sigma(t,\cdot)\|_{L^2}^2 \leqslant \varepsilon\big(\|\nabla\sigma(t,\cdot)\|_{L^2}^2 + \|\sigma\|_{L^2}^2\big)\\ &+\Big(C_\varepsilon + C\big(1 + \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^2}^2\big)\big(1 + \|\nabla\psi_1\|_{L^2}^6 + \|\nabla\psi_2\|_{L^2}^6\big)\Big)\|\sigma\|_{L^2}^2\,.\end{aligned} \tag{14.40}$$

By virtue of (11.18) and (11.19), for each $T > 0$ the following inclusions hold:

$$\psi_1 \in L^\infty(0,T;L^2(G)), \quad \nabla\psi_i \in L^\infty(0,T;L^2(G)), \quad \Delta\psi_i \in L^2(0,T;L^2(G)) \tag{14.41}$$

for $i = 1, 2$. Since the $\psi_i$ satisfy the boundary condition (2.2), we have, by virtue of the estimates for the solution of the elliptic boundary value problem,

[5] The last condition follows from the assumptions (3.16) and (3.17).

$$\|\psi_i\|^2_{H^2(G)} \leqslant C\big(\|\Delta\psi_i\|^2_{L^2(G)} + \|\nabla\psi_i\|^2_{L^2(G)} + \|\psi_i\|^2_{L^2(G)}\big) \quad \text{for } i = 1, 2\,. \tag{14.42}$$

The bounds (14.41) and (14.42) imply that for each $T > 0$ the following estimate for the expression from the right-hand side of (14.40) holds:

$$\int_0^T \Big(C_\varepsilon + C\big(1 + \|\psi_1\|^2_{H^2} + \|\psi_2\|^2_{H^2}\big)\big(1 + \|\nabla\psi_1\|^6_{L^2} + \|\nabla\psi_2\|^6_{L^2}\big)\Big)\, dt < \infty\,. \tag{14.43}$$

Therefore, moving the term $\varepsilon\|\nabla\sigma\|^2_{L^2}$ from the right-hand side of (14.40) to the left-hand side and applying to the result the Gronwall inequality, we find that $\sigma(t, x) \equiv 0$. □

Lemma 14.1 and Theorem 14.2 imply the following result.

**Theorem 14.3.** *Let the assumptions of Theorem* 14.2 *hold. Then the weak statistical solution $\nu$ of the Ginzburg–Landau equation* (3.22) *is uniquely defined by the initial measure $\mu$ and the Wiener measure $\Lambda$.*

We consider now the case of additive white noise when $d = \dim G = 3$.

**Theorem 14.4.** *Let $d = \dim G = 3$ and $r(\lambda) \equiv \rho_1$, and let $\mathcal{K}_{11}(x, x) \in L^p(G)$ with $p > \frac{3}{2}$, where $\mathcal{K}_{11}(x, y)$ is the correlation f unction for the Wiener measure $\Lambda$. Then for each datum $(\psi_0, W) \in F$ a solution $\psi \in K$ of the problem* (12.1) *and* (2.2) *is unique.* (*Here, $F$ and $K$ are the sets defined in* (13.9) *and* (13.13) *respectively.*)

*Proof.* Taking into account the proof of Theorem 14.2, it is enough to establish the bound (14.40) that follows from (14.39) and the estimates for $|T_j|$, $j = 2, \dots, 9$. Recall that, except for $j = 6$ and 8, estimates for all $|T_j|$ were obtained for $d = \dim G \leqslant 3$. So we have to estimate $|T_6|$ and $|T_8|$. Since $r(\lambda) \equiv$ constant, the equality $\partial_j r^{-1} \equiv 0$ holds and therefore, by (14.17), $T_8 = 0$. By virtue of (14.5), (14.12), and (14.15), we obtain for $r(\lambda) \equiv \rho_1$:

$$|T_6| = \Big|\int_G i\nabla\sigma \cdot A(x)\overline{\sigma}\, dx\Big| \leqslant \varepsilon\|\nabla\sigma\|^2_{L^2(G)} + C_\varepsilon\|\sigma\|^2_{L^2(G)}\,.$$

This complete the proof of estimate (14.40) and the proof of the theorem. □

Lemma 14.1 and Theorem 14.4 imply the following result.

**Theorem 14.5.** *Let the assumptions of Theorem* 14.4 *hold. Then the weak statistical solution $\nu$ of the Ginzburg–Landau equation* (3.22) *is uniquely defined by the initial measure $\mu$ and the Wiener measure $\Lambda$.*

# 15 The Strong Statistical Solution of the Stochastic Ginzburg–Landau Equation

Here, we construct the strong statistical solution, prove its uniqueness, and show that it satisfies not only Equation (12.1), but the problem (3.22), (2.2), and (2.3) as well.

## *15.1 Existence and uniqueness of a strong statistical solution*

Recall that, in Sect. 3, an abstract probability space $(\Omega, \Sigma, m(d\omega))$, a random Wiener process $W : \Omega \to C(0, \infty; L^2(G))$, and a random initial condition $\psi_0 : \Omega \to L^1(G)$ were introduced such that $\psi_0(t, \omega)$ and $W(t, x, \omega)$ are independent. In addition, the Wiener measure $\Lambda(dW)$ is a probability distribution of $W(t, x, \omega)$ and $\mu(d\psi_0)$ is a probability distribution of the initial condition $\psi_0(t, \omega)$. Above, we proved the existence of a weak statistical solution $\nu(\Gamma)$, $\Gamma \in \mathcal{B}(\mathcal{U})$, that satisfies (12.6) with the operator $\mathfrak{A}$ defined in (12.1) and (12.5). Based on this existence theorem, we proved in Theorem 13.3 that there exists an $\mu \times \Lambda$-measurable set $F$, defined in (13.9), such that for each datum $(\psi_0, W) \in F$ there exists a solution $\psi \in K$ of (12.1) (the set $K$ is defined in (13.13)). Moreover, in Theorem 14.2, we proved that this solution $\psi$ is unique in $K$. This means that the operator

$$\mathfrak{A}^{-1} \equiv (L, \gamma_0)^{-1} : F \to K\,, \tag{15.1}$$

where $L$ is defined in (12.1), is uniquely defined. We introduce the set

$$\Omega_0 = \{\omega \in \Omega \,:\, (\psi_0(\cdot, \omega); W(\cdot, \cdot, \omega) \in F\}\,. \tag{15.2}$$

Since, by (13.11), $\mu \times \Lambda(F) = 1$ we obtain

$$m(\Omega_0) = 1\,. \tag{15.3}$$

We define the random function

$$\psi(t, x, \omega) = \begin{cases} (L, \gamma_0)^{-1}\big(\psi_0(\cdot, \omega), W(\cdot, \cdot, \omega)(t, x), & \omega \in \Omega_0, \\ 0, & \omega \in \Omega \setminus \Omega_0\,. \end{cases} \tag{15.4}$$

Analogous to the approach in [44, Chapt. 10, Proposition 4.3], one can prove the measurability of the map

$$\psi : (\Omega, \Sigma) \to (\mathcal{U}, \mathcal{B}(\mathcal{U}))\,. \tag{15.5}$$

The relations (15.4) and (12.6) imply that the weak statistical solution $\nu(d\psi)$ is a probability distribution of the random map (15.4). By definition, the random map (15.4) satisfies (12.1) for $m$-almost all $\omega \in \Omega$. Theorem 14.2 implies that the solution (15.4) and (15.5) is defined uniquely by the random datum $(\psi_0(\cdot,\omega), W(\cdot,\cdot,\omega))$.

Note that the assumption (10.3) on the initial measure $\mu(d\psi_0)$ implies that the initial random value $\psi_0(t,\omega)$ satisfies

$$\int \Big(\|\psi_0\|^2_{L^2(G)} + \|\nabla\psi_0\|_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\Big) m(d\omega) < \infty\,. \tag{15.6}$$

Moreover, Theorem 11.8, (2.5), and (13.1) imply that the following inequalities hold:

$$\begin{gathered}\int\limits_{\mathcal{U}_T} \Big(\|\psi\|^2_{L^\infty(0,T;H^1(G))} + \int\limits_0^T \big(\|\psi\|^2_{H^2(G)} + \|\psi\|^6_{L^6(G)}\big)\,dt\Big) m(d\omega) \\ \leqslant C_T\Big(1 + \int \big(\|\psi_0\|^2_{H^1(G)} + \|\psi_0\|^4_{L^4(G)}\big)\Big) m(d\omega)\end{gathered} \tag{15.7}$$

and

$$\int\limits_{\mathcal{U}_T} \|\psi\|_{C^L(0,T;L^1(G))} m(d\omega) \leqslant C_T\Big(1 + \int\big(\|\psi_0\|^2_{H^1(G)} + \|\psi_0\|^4_{L^4(G)}\big) m(d\omega)\Big)\,. \tag{15.8}$$

Thus, we have proved the following result.

**Theorem 15.1.** *Assume that the random initial value $\psi_0(x,\omega)$ and the Wiener process $W(t,x,\omega)$ are independent and $\psi_0$ satisfies* (15.6)*. Then the definition* (15.2) *and* (15.4) *of the strong statistical solution $\psi(t,x,\omega)$ is correct. $\psi(x,\omega)$ satisfies* (12.1) *for $m$-almost all $\omega$ and, by virtue of this equation, $\psi$ is defined uniquely by the datum $(\psi_0(\cdot,\omega), W(\cdot,\cdot,\omega))$. Moreover, $\psi$ satisfies the bounds* (15.7) *and* (15.8).

## 15.2 On one family of scalar Wiener processes

In order to complete our investigation, we have to prove that the random process (15.4) satisfies the stochastic Ginzburg–Landau equation (3.22) or (what is equivalent) (3.24). To do this, we have to provide some preliminary results.

Since the function $\mathcal{K}(x,y)$ from (3.14) is the kernel of the correlation operator for the complex Wiener process $W(t,x,\omega)$ and this operator is self-adjoint non-negative and of trace-class one, the set of all eigenfunctions $\{e_j(x), j = 1,2,\dots\}$ of this operator composes an orthonormal basis in the complex space $L^2(G)$. Moreover, if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \geqslant 0$ are the corresponding eigenvalues, then the following identity holds:

$$\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}\,. \tag{15.9}$$

We introduce the following family of scalar (complex-valued) Wiener processes:

$$W_j(t,\omega) = \int_G W(t,x,\omega)\overline{e_j(x)}\,dx \quad \text{for } j = 1,2,\ldots. \tag{15.10}$$

Then, evidently,

$$W(t,x,\omega) = \sum_{j=1}^{\infty} W_j(t,\omega)e_j(x)\,. \tag{15.11}$$

Recall that for each random function $f(\omega)$ the following notation is used:

$$Ef = \int_\Omega f(\omega)m(d\omega)\,. \tag{15.12}$$

**Lemma 15.2.** *For the Wiener processes* (15.10) *the following identities hold*:

$$EW_j(t)W_m(s) = 0 \quad \forall\, j,m \in N \tag{15.13}$$

*and*

$$EW_j(t)\overline{W_m(s)} = t \wedge s\; \lambda_m \delta_{jm}\,, \tag{15.14}$$

*where $\delta_{jm}$ is the Kronecker delta symbol.*

*Proof.* To prove (15.13), we substitute (15.11) into (3.11), multiply the resulting inequality by $\overline{e_j(x)e_m(y)}$, and integrate with respect to $x$ and $y$. To prove (15.14), we substitute (15.11) into (3.14) and repeat the steps indicated above. □

As is well-known, (15.13) and (15.14) are equivalent to the independence of $W_j(t)$ and $W_m(s)$ for each $j$ and $m$ and to $W_j(t)$ and $\overline{W_m(s)}$ for $j \neq m$.

Consider now the question of the independence of $\operatorname{Re} W_j(t)$ and $\operatorname{Im} W_m(s)$ that are defined by

$$W_j(t) = \operatorname{Re} W_j(t) + i\operatorname{Im} W_j(t)\,. \tag{15.15}$$

**Lemma 15.3.** *For the Wiener processes* $\operatorname{Re} W_j(t)$ *and* $\operatorname{Im} W_m(s)$ *the following identities hold*:

$$E\,\operatorname{Re} W_j(t)\operatorname{Re} W_m(s) = E\,\operatorname{Im} W_j(t)\operatorname{Im} W_n(s) = \frac{1}{2}t \wedge s\;\delta_{jm}\lambda_m \tag{15.16}$$

*and*

$$E \operatorname{Re} W_j(t) \operatorname{Im} W_m(s) = 0\,. \tag{15.17}$$

*Proof.* Substitution of (15.15) into (15.13) and (15.14) gives

$$\begin{aligned} &E \operatorname{Re} W_j(t) \operatorname{Re} W_m(s) - E \operatorname{Im} W_j(t) \operatorname{Im} W_m(s) \\ &\quad + iE \operatorname{Re} W_j(t) \operatorname{Im} W_m(s) + iE \operatorname{Im} W_j(t) \operatorname{Re} W_m(s) = 0 \end{aligned} \tag{15.18}$$

and

$$\begin{aligned} &E \operatorname{Re} W_j(t) \operatorname{Re} W_m(s) + E \operatorname{Im} W_j(t) \operatorname{Im} W_m(s) \\ &\quad - iE \operatorname{Re} W_j(t) \operatorname{Im} W_m(s) + iE \operatorname{Im} W_j(t) \operatorname{Re} W_m(s) = \lambda_j \, t \wedge s \; \delta_{jm}\,, \end{aligned} \tag{15.19}$$

respectively. In fact, (15.18) and (15.13) are four linear algebraic equations in terms of four unknown quantities. Solving these equations, we obtain (15.16) and (15.17). □

## 15.3 Equation for a strong statistical solution

We are now in a position to prove that the strong statistical solution $\psi(t,x)$, constructed in Sect. 15.1, satisfies the Ginzburg–Landau equation (3.22) or, what is equivalent, (3.24). Here, we understand the Ito integral in (3.24) using the decomposition (15.11):

$$\begin{aligned} &\psi(t,x) + \int_0^t \Big( (i\nabla + A)^2 \psi(s,x) - \psi(s,x) + |\psi(s,x)|^2 \psi(s,x) \Big) ds \\ &\qquad = \sum_{j=1}^{\infty} \int_0^t \widehat{r}[\psi(s,x)] \{ e_j(x) dW_j(t) \} \Big) + \psi_0(x)\,. \end{aligned} \tag{15.20}$$

The integral on $ds$ in (15.20) is understood as a Bochner integral for a function with values in $L^2(G)$. To explain the meaning of the stochastic integral in (15.20), we first write, using (3.20) and (3.21), the identity

$$\begin{aligned} &\int_0^t \widehat{r}[\psi(s,x)] \big\{ e_j(x)\; dW_j(t) \big\} = \operatorname{Re} e_j(x) \int_0^t r(\operatorname{Re} \psi(s,x))\; d\operatorname{Re} W_j(s) \\ &- \operatorname{Im} e_j(x) \int_0^t r(\operatorname{Re} \psi(s,x)\; d\operatorname{Im} W_j(s) \end{aligned}$$

$$+i\big\{\mathrm{Re}\, e_j(s)\int\limits_0^t r(\mathrm{Im}\,\psi(s,x))\; d\mathrm{Im}\, W_j(s)$$
$$+\mathrm{Im}\, e_j(s)\int\limits_0^t r(\mathrm{Im}\,\psi(s,x))\; d\mathrm{Re}\, W_j(s)\big\}\,. \tag{15.21}$$

The stochastic integrals on the right-hand side of (15.21) are understood in the usual classical sense (see, for example, [26]) for each fixed $x \in G$ because $\psi(s,x) \in L^2(0,T;H^2(G)) \subset L^2(0,T;C(\overline{G}))$.

Multiplying both parts of (15.21) by an arbitrary $\overline{v(x,\omega)} \in L^2(G \times \Omega)$, integrating on $x$ over $G$, squaring, applying Doob's inequality (see [26, p. 174], and taking into account (15.16), we obtain, for each $T > 0$,

$$E \sup_{t\in[0,T]} \left| \int\limits_G \overline{v(x)} \int\limits_0^t \widehat{r}[\psi(s,x)]\big\{e_j(x) dW_j(s)\big\} dx \right|^2$$
$$= E \sup_{t\in[0,T]} \left| \int\limits_0^t \int\limits_G \overline{v}(x)\widehat{r}[\psi(s,x)]\big\{e_j(x) dx dW_j(s)\big\} \right|^2 \tag{15.22}$$
$$\leqslant C\lambda_j E \int\limits_0^T \Big( \int\limits_G |v(x)|\; |e_j(x)|\; |r[\psi(s,x)]|\; dx\Big)^2 ds$$
$$\leqslant C\lambda_j E\|v\|^2_{L^2}\big(1 + E\|\psi\|^2_{L^2(0,T;H^2(G))}\big)\,,$$

where $C$ does not depend on $j$. Since $\sum\limits_j \lambda_j \leqslant C$, the inequality (15.22) proves that the series on the right-hand side of (15.20) converges weakly in $L^2(G \times \Omega)$. Thus, all terms in (15.20) are well-defined.

**Theorem 15.4.** *Let the conditions of Theorem* 15.1 *be fulfilled. Then the random process* $\psi(t,x,\omega)$ *defined in* (15.4) *satisfies Equation* (15.20).

*Proof.* We apply the Ito formula (see [26, Chapt. 6, Sect. 5]) [6] to the stochastic integral $S[\psi(t,x)]$ that is defined by (12.1). Note that, by (15.11), the stochastic integral from (12.1) can be rewritten as follows

[6] In [26], the Ito formula has been proved for a finite-dimensional vector-valued Wiener process $\mathbf{W}(t) = (W_j(t)), j = 1,\dots,n$. In order to extend this proof to a stochastic integral with an infinite-dimensional vector-valued Wiener processes as in (15.20), it is enough to apply the arguments that were used above to explain the meaning of the stochastic integral in (15.20).

$$dS[\psi(t,x)] + \widehat{r^{-1}}[\psi(t,x)]\big\{(i\nabla + A)^2\psi(t,x) - \psi(t,x) + |\psi|^2\psi(t,x)\big\}$$
$$+\frac{1}{2}\widehat{r'}[\psi]\mathcal{K}_{11}(x,x) = \sum_{j=1}^{\infty} e_j(x)dW_j(t)\,. \tag{15.23}$$

By virtue of (15.16) and (15.17), for the calculation of $dW_j(t)dW_m(t)$ in the Ito formula, we can use the identities

$$d\mathrm{Re}\,W_j d\mathrm{Re}\,W_m = d\mathrm{Im}\,W_j d\mathrm{Im}\,W_m = \frac{1}{2}\lambda_j\delta_{jm}dt \tag{15.24}$$

and

$$d\mathrm{Re}\,W_j d\mathrm{Im}\,W_m = d\mathrm{Re}\,W_j dt = d\mathrm{Im}\,W_m dt = 0 \quad \forall\, m, j\,. \tag{15.25}$$

Recall that the functions $r(\lambda)$, $S(\lambda)$, and $R(\lambda)$ are defined in (3.19), (7.7), and (7.23) respectively. We apply Ito's formula to the functional $\int_G R[S(t,x)]\cdot \overline{v(x)}\,dx$, where $v(x) \in L^2(G)$. We have

$$d\int R[S[\psi(t,x)]]\cdot\overline{v(x)}\,dx = \int \widehat{R'}[S[\psi]]\{dS\}\cdot\overline{v(x)}\,dx$$
$$+\frac{1}{2}\widehat{R''}[S[\psi]]\{dS,dS\}\overline{v(x)}\,dx\,. \tag{15.26}$$

By (7.7) and (7.23), $R'(S(\lambda)) = r(\lambda)$. Using this and (15.23), we obtain

$$\int \widehat{R'}[S[\psi]]\{dS\}\cdot\overline{v(x)}\,dx = \int \widehat{r}[\psi]\{dS\}\cdot\overline{v}\,dx$$
$$= -\int_G \widehat{r}[\psi]\big\{\widehat{r^{-1}}[\psi]\big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi\big)$$
$$+\frac{1}{2}r'[\psi]\mathcal{K}_{11}(x,x)\big\}\cdot\overline{v(x)}\,dxdt + \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\} \tag{15.27}$$
$$= -\int_G \big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi$$
$$+\frac{1}{2}\widehat{r}[\psi]\{r'[\psi]\}\mathcal{K}_{11}(x,x)\big)\overline{v(x)}\,dt + \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\}\,.$$

This term can be rewritten by using (3.20) and (3.21) as

$$\int R'[S[\psi]]\{dS\}\overline{v(x)}\,dx = \int \widehat{r}[\psi]\{dS\}\overline{v}\,dx$$
$$= \int \Big(r(\operatorname{Re}\psi)d\operatorname{Re}S + ir(\operatorname{Im}\psi)d\operatorname{Im}S\Big)\overline{v}\,dx\,. \tag{15.28}$$

By virtue of (15.23)–(15.25) and (15.28), we can rewrite the second term on the right-hand side of (15.26) as

$$\frac{1}{2}\int \widehat{R''}[S[\psi]]\{dS,dS\}\overline{v(x)}\,dx$$
$$= \frac{1}{2}\int\Big(\partial_{\operatorname{Re}S}r\big(R(\operatorname{Re}S)\big)d\operatorname{Re}S d\operatorname{Re}S + i\partial_{\operatorname{Im}S}r\big(R(\operatorname{Im}S)\big)\ d\operatorname{Im}S\ d\operatorname{Im}S\Big)\overline{v}\,dx$$
$$= \frac{1}{2}\int\Big(r'(\operatorname{Re}\psi)r(\operatorname{Re}\psi)d\operatorname{Re}W\ d\operatorname{Re}W$$
$$+ir'(\operatorname{Im}\psi)r(\operatorname{Im}\psi)d\operatorname{Im}W\ d\operatorname{Im}W\Big)\overline{v}\,dx$$
$$= \frac{1}{2}\int\Big(r'(\operatorname{Re}\psi)r(\operatorname{Re}\psi)\sum_j\big(d\operatorname{Re}W_j\operatorname{Re}e_j - d\operatorname{Im}W_J\operatorname{Im}e_j\big)$$
$$\cdot\Big(\sum_m\big(d\operatorname{Re}W_m\operatorname{Re}e_m - d\operatorname{Im}W_m\operatorname{Im}e_m\big)\Big)$$
$$+ir'(\operatorname{Im}\psi)r(\operatorname{Im}\psi)\Big(\sum_j d\operatorname{Im}W_j\operatorname{Re}e_j + d\operatorname{Re}W_j\operatorname{Im}e_j\Big)$$
$$\cdot\Big(\sum_m d\operatorname{Im}W_m\operatorname{Re}e_m + d\operatorname{Re}W_m\operatorname{Im}e_m\Big)\Big)\overline{v(x)}\,dx$$
$$= \frac{1}{2}\int\Big(r'(\operatorname{Re}\psi)r(\operatorname{Re}\psi)\frac{1}{2}\sum_j\lambda_j|e_j(x)|^2$$
$$+ir'(\operatorname{Im}\psi)r(\operatorname{Im}\psi)\frac{1}{2}\sum_j\lambda_j|e_j(x)|^2\Big)\overline{v(x)}\,dxdt \equiv T\,. \tag{15.29}$$

By (15.9) and (3.14), $\sum_j \lambda_j|e_j(x)|^2 = 2\mathcal{K}_{11}(x,x)$ and therefore the right-hand side of (15.29) is equal to the expression

$$T = \int \widehat{r'}[\psi]\{r[\psi]\mathcal{K}_{11}(x,x)\}\overline{v(x)}\,dxdt\,. \tag{15.30}$$

Taking into account that, on the left-hand side of (15.26), $R[S[\psi(t,x)]] = \psi(t,x)$, we obtain from (15.26), (15.27), (15.29), and (15.30) the final formula

$$d\int\limits_G \psi(t,x)\overline{v(x)}\,dx + \int\limits_G \big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi\big)\overline{v(x)}\,dx = \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\}\overline{v}\,dx\,. \tag{15.31}$$

This equality holds for each $v(x) \in L^2(G)$. Clearly, this equality is equivalent to

$$d\psi(t,x) + \big\{(i\nabla + A)^2\psi(t,x) - \psi(t,x) + |\psi(t,x)|^2\psi(t,x)\big\}\,dt = \widehat{r}[\psi]\Big\{\sum_{j=1}^{\infty} e_j(x)dW_j(t)\Big\} \tag{15.32}$$

and (15.32) is equivalent to (15.20). □

**Acknowledgment.** The first author was partially supported by the School of Computational Science at Florida State University during visits in 2003 and 2007 and also partially supported by Russian Academy of Sciences Program "Theoretical problems of modern mathematics," project "Optimization of Numerical Algorithms of Mathematical Physics Problems" and by the Russian Foundation for Basic Research (grant no. 04-01-0066).

## References

1. Bardeen, J., Cooper, L., Schrieffer, J.: Theory of superconductivity. Phys. Rev. **108**, 1175-1204 (1957)
2. Bensoussan, A., Temam, R.: Equations aux derivees partielle stochastiques non linearies (1) Isr. J. Math. **11**, 95-129 (1972)
3. Bensoussan, A., Temam, R.: Equations stochastiques du type Navier–Stokes. J. Funct. Ana. **13**, 195-222 (1973)
4. Chapman, J., Du, Q., Gunzburger, M.: A Ginzburg–Landau type model of superconducting/normal junctions including Josephson junctions. Europ. J. Appl. Math. **6**, 97-114 (1995)
5. Chapman, J., Du, Q., Gunzburger, M.: A model for variable thickness superconducting thin films. Z. Angew. Math. Phys. **47**, 410-431 (1996)
6. Chapman, J., Du, Q., Gunzburger, M., Peterson, J.: Simplified Ginzburg–Landau type models of superconductivity in the high kappa, high field limit. Adv. Math. Sci. Appl. **5**, 193-218 (1995)
7. Chapman, J., Howinson, S., Ockendon, J.: Macroscopic models for superconductivity. SIAM Review **34**, 529-560 (1992)

8. Daletskii, Yu.: Infinite dimensional elliptic operators and connected with them parabolic equations (in Russian). Uspekhi Mat. Nauk **22**, 3-54 (1967)
9. Deang, J., Du, Q., Gunzburger, M.: Stochastic dynamics of Ginzburg–Landau vortices in superconductors. Phys. Rev. B **64**, 052506 (2001)
10. Deang, J., Du, Q., Gunzburger, M.: Modeling and computation of random thermal fluctuations and material defects in the Ginzburg–Landau model for superconductivity. J. Comput. Phys. **181**, 45-67 (2002)
11. Dorsey, A., Huang, M., Fisher M.: Dynamics of the normal to superconducting vortex-glass transition: Mean-field theory and fluctuations. Phys. Rev. B **45**, 523-526 (1992)
12. Du, Q., Gray, P.: High-kappa limit of the time dependent GinzburgLandau model for superconductivity. SIAM J. Appl. Math. **56**, 1060-1093 (1996)
13. Du, Q., Gunzburger, M., Peterson, J.: Analysis and approximation of the Ginzburg–Landau model of superconductivity. SIAM Review **34**, 54-81 (1992)
14. E, W., Mattingly, J., Sinai, Y.: Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. Commun. Math. Phys. **224**, 83-106 (2001)
15. Filippov, A., Radievsky, A., Zelster, A.: Nucleation at the fluctuation induced first order phase transition to superconductivity. Phy. Lett. A **192**, 131-136 (1994)
16. Flandoli, F., Maslowski, B.: Ergodicity of the 2D Navier-Stokes equation under random perturbations. Commun. Math. Phys. **172**, 119-141 (1995)
17. Fursikov, A.: Optimal Control of Distributed Systems. Theory and Applications. Am. Math. Soc., Providence, RI (1999)
18. Gikhman, I., Skorokhod, A.: Introduction to the Theory of Random Processes. Dover, New York (1969)
19. Gikhman, I., Skorokhod, A.: The Theory of Stochastic Processes. Springer, Berlin (1974)
20. Ginzburg, V., Landau, L.: On the theory of superconductivity (in Russian). Zh. Eksperim. Teor. Fiz. **20**, 1064-1082 (1950); English transl.: In: ter Haar, D. (ed.) Men of Physics: L. D. Landau, pp. 138-167. Pergamon, Oxford (1965)
21. Gor'kov, L.: Microscopic derivation of the Ginzburg–Landau equations in the theory of superconductivity (in Russian). Zh. Eksperim. Teor. Fiz. **36**, 1918-1923 (1959); English transl.: Sov. Phys.–JETP **9**, 1364-1367 (1959)
22. Hohenberg, P., Halperin, B.: Theory of dynamic critical phenomena. Rev. Mod. Phy. **49**, 435-479 (1977)
23. Kamppeter, T., Mertens, F., Moro, E., Sanchez, A., Bishop, A.: Stochastic vortex dynamics in two-dimensional easy-plane ferromagnets: Multiplicative versus additive noise. Phys. Rev. B **59**, pp.11 349-11 357 (1991)
24. Krylov, N.: Introduction to the Theory of Diffusion Processes. Am. Math. Soc., Providence, RI (1996)
25. Krylov, N.: SPDEs in $L_q((0,\tau], L_p)$ spaces. Electron. J. Probab. **5**, 1-29 (2000)
26. Krylov, N.: Introduction to the Theory of Random Processes. Am. Math. Soc., Providence, RI (2002)
27. Krylov, N., Rozovskii, B.: Stochastic evolution equation. J. Sov. Math. **16**, 1233-1277 (1981)
28. Kuksin, S.: Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions. Zurich Lectures in Advanced Mathematics, European Math. Soc. (EMS), Zurich (2006)
29. Kuksin, S., Shirikyan, A.: Stochastic dissipative PDE's and Gibbs measures. Commun. Math. Phys. **213**, 291-330 (2000)
30. Larkin, A.: Effect of inhomogeneities on the structure of the mixed state of superconductors. Sov. Phys. JETP **31**, 784-786 (1970)
31. Lions, J.-L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications I. Dunod, Paris (1968)

32. Mikulevicius, R., Rozovskii, B.: Stochastic Navier–Stokes Equations for Turbulent Flows. Preprint (2003)
33. Pardoux, E.: Sur le equations aux derivees partielles stochastiques monotones. C. R. Acad. Sci. **275**, A101-A103 (1972)
34. Rozovskii, B.: Stochastic evolution systems. Kluwer, Dordrecht (1990)
35. Sasik, R. , Bettencourt, L., Habib, S.: Thermal vortex motion in a two-dimensional condensate. Phys. Rev. B **62**, 1238-1243 (2000)
36. Shirikyan, A.: Ergodicity for a class of Markov processes and applications to randomly forced PDE's. I. Russ. J. Math. Phys. **12**, 81-96 (2005)
37. Shirikyan, A.: Ergodicity for a class of Markov processes and applications to randomly forced PDE's II. Discrete Contin. Dyn. Syst. Ser. B **6**, 911-926 (2006)
38. Shirikyan, A.: Exponential mixing for randomly forced PDE's: method of coupling. In: Bardos, C., Fursikov, A. Eds. Instability of Models Connected with Fluid Flows II, pp. 155-188. Springer, New York (2008)
39. Skocpol, W., Tinkham, M.: Fluctuations near superconducting phase transitions. Rep. Progr. Phys. **38**, 1049-1097 (1975)
40. Sobolev, S.L.: Some Applications of Functional Analysis in Mathematical Physics (in Russian). 1st ed. Leningrad State Univ., Leningrad (1950); 3rd ed. Nauka, Moscow (1988); English transl. of the 1st Ed.: Am. Math. Soc., Providence, RI (1963); English transl. of the 3rd Ed. with comments by V. P. Palamodov: Am. Math. Soc., Providence, RI (1991)
41. Tinkham, M.: Introduction to Superconductivity. McGraw-Hill, New York (1975)
42. Troy, R., Dorsey, A.: Transport properties and fluctuations in type-II superconductors near Hc2. Phys. Rev. B **47**, 2715-2724 (1993)
43. Ullah, S., Dorsey, A.: The effect of fluctuations on the transport properties of type-II superconductors in a magnetic field. Phys. Rev. B **44**, 262-273 (1991)
44. Vishik, M., Fursikov, A.: Mathematical Problems of Statistical Hydromechanics. Kluwer, Boston (1988)
45. Vishik, M., Komech, A,: On solvability of the Cauchy problem for the direct Kolmogorov equation corresponding to the stochastic Navier–Stokes type equation (in Russian). In: Complex Analysis and its Applications, pp. 126-136. Nauka, Moscow (1978)

# Carleman Estimates with Second Large Parameter for Second Order Operators

Victor Isakov and Nanhee Kim

**Abstract** We prove Carleman type estimates with two large parameters for general linear partial differential operators of second order. Using the second large parameter, from results for scalar equations we derive Carleman estimates for dynamical Lamé system with residual stress. These estimates are used to prove the Hölder and Lipschitz stability for the continuation of solutions under pseudoconvexity assumptions. So, the first uniqueness and stability of the continuation results are established for an important anisotropic system of elasticity without the assumption that this anisotropic system is close to an isotropic system.

## 1 Introduction

While the main idea to use special exponential weight in energy integrals belongs to Carleman, the language and technique of this paper heavily use Sobolev spaces and their properties. Also the more general idea of exploiting various concepts of energy in mathematical physics (in particular, in the elasticity theory) leads naturally to weak solutions and it was pioneered by Sobolev in the 1930s. So our work is deeply influenced by discoveries of Sobolev.

We consider the general partial differential operator of second order

$$A = \sum_{j,k=1}^{n} a^{jk}\partial_j\partial_k + \sum b^j\partial_j + c$$

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series.* 
doi: 10.1007/978-0-387-85652-0, 

in a bounded domain $\Omega$ of the space $\mathbb{R}^n$ with real-valued coefficients $a^{jk} \in C^1(\overline{\Omega})$, $b^j, c \in L_\infty(\Omega)$. The principal symbol of this operator is

$$A(x;\zeta) = \sum a^{jk}(x)\zeta_j\zeta_k. \tag{1.1}$$

We use the following conventions and notation: sums are taken over repeated indices $j, k, l, m = 1, \ldots, n$; $\partial = (\partial_1, \ldots, \partial_n)$, $D = -i\partial$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex with integer components, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$, $D^\alpha$ and $\partial^\alpha$ are defined similarly; $\nu$ is the outward normal to the boundary of a domain; $C$ denotes generic constants (different at different places) depending only on the operators $A$ or $\mathbf{A}_R$, the function $\psi$, and the domain $\Omega$. The dependence on additional parameters will be indicated. We recall that

$$\|u\|_{(k)}(\Omega) = \Big( \sum_{|\alpha| \leqslant k} \int_\Omega |\partial^\alpha u|^2 \Big)^{\frac{1}{2}}$$

is the norm in the Sobolev space $H_{(k)}(\Omega)$ and $\|\|_2 = \|\|_{(0)}$ is the $L^2$-norm.

A function $\psi$ is called *pseudoconvex* on $\overline{\Omega}$ with respect to $A$ if $\psi \in C^2(\overline{\Omega})$, $A(x, \nabla\psi(x)) \neq 0$, $x \in \overline{\Omega}$, and

$$\sum \partial_j \partial_k \psi(x) \frac{\partial A}{\partial \zeta_j} \frac{\partial A}{\partial \zeta_k}(x;\xi) + \sum \Big( \frac{\partial A}{\partial \zeta_k} \partial_k \frac{\partial A}{\partial \zeta_j} - \partial_k A \frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k} \Big) \partial_j \psi(x,\xi) \geqslant K|\xi|^2 \tag{1.2}$$

for some positive constant $K$, any $\xi \in R^n$, and any point $x$ of $\overline{\Omega}$ provided that

$$A(x;\xi) = 0, \quad \sum \frac{\partial A}{\partial \zeta_j}(x,\xi)\partial_j\psi(x) = 0. \tag{1.3}$$

We use the weight function

$$\varphi = e^{\gamma\psi}. \tag{1.4}$$

Let $\sigma = \gamma\tau\varphi$ and $\Omega_\varepsilon = \Omega \cap \{\psi(x) > \varepsilon\}$.

**Theorem 1.1.** *Let $\psi$ be pseudoconvex with respect to $A$ in $\overline{\Omega}$. Then there are constants $C$ and $C_0(\gamma)$ such that*

$$\int_\Omega \sigma^{3-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha u|^2 \leqslant C \int_\Omega e^{2\tau\varphi} |Au|^2 \tag{1.5}$$

*for all $u \in C_0^2(\Omega)$, $|\alpha| \leqslant 1$, $C < \gamma$, and $C_0(\gamma) < \tau$.*

The weighted energy type estimates with large parameter $\tau$ were first introduced by Carleman in 1939 to prove the first uniqueness of continuation results for elliptic systems with nonanalytic coefficients in the plane. The idea of Carleman turned out to be extremely fruitful; in the 1950-70s, this idea was applied to many important partial differential equations. Hörmander [6] linked it to the pseudoconvexity condition for the theory of functions of several complex variables and to energy estimates for general hyperbolic equations. At present, there are several interesting (and, in some cases, complete) results on Carleman estimates and the uniqueness of continuation for second order equations, including elliptic, parabolic, Schrödinger type, and hyperbolic equations [10, 15].

However, systems of partial differential equations still remain a serious challenge. The only available general result is the celebrated theorem of Calderon in 1958 which is applicable mainly to some elliptic systems. There is a progress for the classical dynamical isotropic Maxwell and elasticity systems [5, 8]. The uniqueness of continuation results for some anisotropic systems (including thermoelasticity system) was first obtained by Albano and Tataru [1] and Isakov [9]. In these papers, it was crucial to use Carleman type estimates with two large parameters (1.5) first introduced and applied to the classical elasticity system in [8]. In [3], Theorem 1.1 (for $C^\infty$-coefficients) was stated without proof, and, in [4], there are not complete proofs for isotropic hyperbolic equations.

As an important application of Theorem 1.1, we consider an elasticity system with residual stress $R$ [11, 12, 16]. This system is anisotropic. At present, there are results on the uniqueness of continuation and identification of its coefficients under the assumption that the residual stress is "small" (without a quantitative estimate of smalnness). In [17], there are theorems about the uniqueness of identification for some coefficients of the residual stress under quite complicated conditions and from all possible boundary data. We derive the global uniqueness of continuation in $\Omega_0 \subset \Omega$ under some pseudoconvexity conditions on a weight function $\psi$ defining $\Omega_0$. In Theorems 1.2–1.4 below, $x \in \mathbb{R}^3$ and $(x,t) \in \Omega \subset \mathbb{R}^4$. The residual stress is modeled by a symmetric second rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^2(\overline{\Omega})$ which is divergence free: $\nabla \cdot R = 0$. We denote by $\mathbf{u}(x,t) = (u_1,u_2,u_3)^\top : \Omega \to \mathbb{R}^3$ the displacement vector in $\Omega$ and introduce the operator of linear elasticity with the residual stress

$$\mathbf{A}_R\mathbf{u} = \rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\operatorname{div}\,\mathbf{u}) - \nabla\lambda\operatorname{div}\,\mathbf{u} - 2\varepsilon(\mathbf{u})\nabla\mu - \operatorname{div}\,((\nabla\mathbf{u})R), \tag{1.6}$$

where $\rho \in C^1(\overline{\Omega})$ is the density and $\lambda, \mu \in C^2(\overline{\Omega})$ are the Lamé parameters depending only on $x$, $\varepsilon(\mathbf{u}) = (\frac{1}{2}(\partial_i u_j + \partial_j u_i))$. Let

$$\Box(\mu; R) = \partial_t^2 - \sum_{jk} \frac{\mu\delta_{jk} + r_{jk}}{\rho}\partial_j\partial_k$$

and $\sigma = \tau\gamma\varphi$.

**Theorem 1.2.** *Let $\psi$ be pseudoconvex with respect to $\Box(\mu;R);\Box(\lambda+2\mu;R)$ in $\overline{\Omega}$. Then there are constants $C$ and $C_0(\gamma)$ such that*

$$\int\limits_{\Omega}(\sigma(|\nabla_{x,t}\mathbf{u}|^2+|\nabla_{x,t}\operatorname{div}\mathbf{u}|^2+|\nabla_{x,t}\operatorname{curl}\mathbf{u}|^2)$$

$$+\sigma^3(|\mathbf{u}|^2+|\operatorname{div}\mathbf{u}|^2+|\operatorname{curl}\mathbf{u}|^2))e^{2\tau\varphi}$$

$$\leqslant C\int\limits_{\Omega}(|\mathbf{A}_R\mathbf{u}|^2+|\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} \tag{1.7}$$

*for all $\mathbf{u}\in H^0_{(3)}(\Omega)$, $C<\gamma$, $C_0<\tau$.*

In [11], this result was obtained for "small" $R$.

We consider the Cauchy problem

$$\mathbf{A}_R\mathbf{u}=\mathbf{f}\quad\text{in}\quad\Omega,\ \mathbf{u}=\mathbf{g}_0,\quad\partial_\nu\mathbf{u}=\mathbf{g}_1\quad\text{on}\quad\Gamma\subset\partial\Omega, \tag{1.8}$$

where $\Gamma\in C^3$. By standard arguments [10, Sect. 3.2], the Carleman estimate in Theorem 1.2 implies the following conditional Hölder stability estimate for (1.8) in $\Omega_\delta$ (and, consequently, the uniqueness in $\Omega_0$).

**Theorem 1.3.** *Suppose that all the coefficients $\lambda$, $\mu$, $\rho$, $R$ are in $C^2(\overline{\Omega})$. Let $\psi$ be pseudoconvex with respect to $\Box(\mu;R)$; $\Box(\lambda+2\mu;R)$ in $\overline{\Omega}_0$. Assume that $\overline{\Omega}_0\subset\Omega\cup\Gamma$. Then there exist $C(\delta),\kappa(\delta)\in(0,1)$ such that for a solution $\mathbf{u}\in H_{(3)}(\Omega)$ to (1.8)*

$$\|\mathbf{u}\|_{(2)}(\Omega_\delta)\leqslant C(F+M^{1-\kappa}F^\kappa), \tag{1.9}$$

*where*

$$F=\|\mathbf{f}\|_{(1)}(\Omega_0)+\|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma)+\|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma),\quad M=\|\mathbf{u}\|_{(2)}(\Omega).$$

In Theorem 1.4, we assume that $\Omega=G\times(-T,T)$ and the system (1.8) is $t$-hyperbolic. Using the known theory [2], one can derive the sufficient condition

$$0\leqslant\lambda,\quad 0<2\mu I_3+R\ \text{ on }\ \overline{\Omega}.$$

This condition is satisfied when any eigenvalue of the matrix $R$ is strictly greater than $-2\mu$. Such a situation happens when, for example, $\sum\limits_{i,j=1}^{3}r_{ij}^2<4\mu^2$ on $\overline{\Omega}$. We use the conventional energy integral

$$E(t;\mathbf{u})=\int\limits_G(|\partial_t\mathbf{u}|^2+|\nabla\mathbf{u}|^2+|\mathbf{u}|^2)(,t).$$

**Theorem 1.4.** *Suppose that* $\lambda$, $\mu$, $\rho$, $R$ *are in* $C^2(\overline{\Omega})$. *Let* $\psi$ *be pseudoconvex with respect to* $\square(\mu; R)$; $\square(\lambda + 2\mu; R)$ *in* $\overline{\Omega}$. *Assume that*

$$\psi < 0 \text{ on } \overline{G} \times \{-T, T\}, \quad 0 < \psi \text{ on } G \times \{0\}.$$

*Then there exists* $C$ *such that for a solution* $\mathbf{u} \in H_{(3)}(\Omega)$ *to* (1.8)

$$E(t; \mathbf{u}) + E(t; \nabla \mathbf{u}) \leqslant C(\|\mathbf{f}\|_{(1)}(\Omega) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)). \qquad (1.10)$$

The paper is organized as follows. In Sect. 2, we describe two kinds of known sufficient conditions of pseudoconvexity for two kinds of functions $\psi$ with respect to a general anisotropic hyperbolic operator $A$. Also, we give a simple new condition of pseudoconvexity in the case, where $R$ is small relative to constants $\rho$, $\mu$, $\lambda$, and explicitly describe this smallness condition. Section 3 is central. Here, we prove Theorem 1.1 by using an explicit form of pseudoconvexity conditions for second order operators such that one can trace the dependence on the second large parameter $\gamma$. The crucial part of the proof is Lemma 3.2 which gives a bound on the symbol of a differential quadratic form. To find a suitable form of this bound was a decisive step in deriving Theorem 1.1. In the remaining part of Sect. 3, we complete the proof by applying the standard Fourier analysis methods augmented by proper localization and the use of large parameter $\tau$. In Sects. 4–6, we modify methods in [10, Chapt. 3], [11], and [12] to derive from Theorem 1.1 the Carleman estimates, as well as local (Hölder type) and global (Lipschitz type) stability estimates for the lateral Cauchy problems for the system (1.6). Finally, we discuss open problems and their possible solutions.

## 2 Pseudoconvexity Condition

It is not easy to find pseudoconvex functions $\psi$ with respect to a general anisotropic operator, in particular, the hyperbolic operator $A = \partial_t^2 - \sum_{j,k=1}^{n} a^{jk}\partial_j\partial_k$. In the isotropic case, explicit and verifiable conditions for $\psi(x,t) = |x - \beta|^2 - \theta^2 t^2$ were found by Isakov in 1980 and their simplifiications were given in [10, Sect. 3.4]. In the case of general hyperbolic equations, Khaidarov [14] showed that the same function $\psi$ is pseudoconvex if the propagation speed determined by $A$ is monotone in a certain direction. The most suitable choice of $\psi$ is $\psi(x,t) = d^2(x, \beta) - \theta^2 t^2$, where $d$ is the distance in the Riemannian metrics determined by the spatial part of $A$. This choice leads, in many cases, to an exact description of the uniqueness domain. Lasiecka, Triggiani, and Yao [15] showed that this function is indeed pseudoconvex when $d^2$ is convex in the Riemannian metric. Romanov [18] gave a simple independent proof and emphasized that the negativity of sectional curvatures is sufficient.

The known conditions of pseudoconvexity in the anisotropic case are hard to verify. For example, the conditions in [15, 18] impose restrictions on the second partial derivatives of $a^{jk}$. In applications, residual stress is relatively small [16]. Motivated by these reasons, we give a simple sufficient condition of pseudoconvexity for $R$, where the "smallness" of $R$ is explicit, contrary to the conditions in [11, 12].

We use the matrix norm

$$\|R\| = \Big( \sum_{j,k=1}^{3} r_{jk}^2 \Big)^{\frac{1}{2}}.$$

**Lemma 2.1.** *Let $\theta$, $s$ be some numbers, $\beta \in \mathbf{R}^n$, $\rho$, $\lambda$, $\mu$ constants, Assume that a matrix $R$ is symmetric positive at any point of $\Omega$ and*

$$2\mu\rho\theta^2 + 3\|R + \mu I\| \|\nabla R\| |x - \beta| < 2\mu^2 \text{ on } \Omega. \tag{2.1}$$

*Let*

$$\theta^2 < \frac{\mu}{\rho} \tag{2.2}$$

*Then the function $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2 - s^2$ is pseudoconvex with respect to the anisotropic wave operator $A = \Box(\mu; R)$ in $\overline{\Omega} \cap \{|x-\beta|^2 > \theta^2 t^2\}$.*

*Proof.* By definition, we need the positivity of the quadratic form

$$\mathcal{H} = \sum_{j,k=0}^{n} \partial_j \partial_k \psi \frac{\partial A}{\partial \xi_j} \frac{\partial A}{\partial \xi_k} + \sum_{j,k=0}^{n} \Big( \Big( \partial_k \frac{\partial A}{\partial \xi_j} \Big) \frac{\partial A}{\partial \xi_k} - \partial_k A \frac{\partial^2 A}{\partial \xi_j \partial \xi_k} \Big) \partial_j \psi.$$

A direct calculation with

$$A(x,\zeta) = \zeta_0^2 - \frac{\mu}{\rho} \zeta \cdot \zeta - \sum_{j,k=1}^{n} \frac{r_{jk}}{\rho} \zeta_j \zeta_k$$

yields

$$\begin{aligned}
\mathcal{H} = & -8\theta^2 \xi_0^2 + 8 \sum_{j=1}^{n} \Big( \frac{1}{\rho} \Big( \sum_{k=1}^{n} r_{jk} \xi_k + \mu \xi_j \Big) \Big)^2 \\
& + \sum_{j,k=1}^{n} \Big\{ \Big( -\frac{2}{\rho} \sum_{l=1}^{n} \partial_k r_{jl} \xi_l \Big) \Big( -\frac{2}{\rho} \Big( \sum_{m=1}^{n} r_{km} \xi_m + \mu \xi_k \Big) \Big) (2(x-\beta)_j) \Big\} \\
& - \sum_{j,k=1}^{3} \Big\{ \Big( -\frac{1}{\rho} \sum_{l,m=1}^{3} \partial_k r_{lm} \xi_l \xi_m \Big) \Big( -\frac{2}{\rho} (r_{jk} + \mu \delta_{jk}) \Big) (2(x-\beta)_j) \Big\}
\end{aligned}$$

$$= -\frac{8}{\rho}\mu\theta^2|\xi|^2 - \frac{8}{\rho}\theta^2 \sum_{j,k=1}^{n} r_{jk}\xi_j\xi_k + \frac{8}{\rho^2}\sum_{j=1}^{n}\Big(\Big(\sum_{k=1}^{n} r_{jk}\xi_k\Big)^2$$

$$+ 2\mu\xi_j\Big(\sum_{k=1}^{n} r_{jk}\xi_k\Big) + \mu^2\xi_j^2\Big)$$

$$+ \frac{8}{\rho^2}\sum_{j,k=1}^{n}\Big\{\Big(\sum_{l=1}^{n}\partial_k r_{jl}\xi_l\Big)\Big(\sum_{m=1}^{n} r_{km}\xi_m + \mu\xi_k\Big)((x-\beta)_j)\Big\}$$

$$- \frac{4}{\rho^2}\sum_{j,k=1}^{n}\Big\{\Big(\sum_{l,m=1}^{n}\partial_k r_{lm}\xi_l\xi_m\Big)\Big(r_{jk} + \mu\delta_{jk}\Big)((x-\beta)_j)\Big\}.$$

Hence

$$\mathcal{H} \geqslant -\frac{8}{\rho}\mu\theta^2|\xi|^2 - \frac{8}{\rho}\theta^2 \sum_{j,k=1}^{n} r_{jk}\xi_j\xi_k$$

$$+ \frac{8}{\rho^2}\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} r_{jk}\xi_k)^2 + \frac{16}{\rho^2}\mu\sum_{j,k=1}^{n} r_{jk}\xi_j\xi_k\Big) + \frac{8}{\rho^2}\mu^2|\xi|^2$$

$$- \frac{8}{\rho^2}\Big|\sum_{j,k=1}^{n}\Big\{\Big(\sum_{l=1}^{n}\partial_k r_{jl}\xi_l\Big)\Big(\sum_{m=1}^{n} r_{km}\xi_m + \mu\xi_k\Big)((x-\beta)_j)\Big\}\Big|$$

$$- \frac{4}{\rho^2}\Big|\sum_{j,k=1}^{n}\Big\{\Big(\sum_{l,m=1}^{n}\partial_k r_{lm}\xi_l\xi_m\Big)\Big(r_{jk} + \mu\delta_{jk}\Big)((x-\beta)_j)\Big\}\Big|$$

$$\geqslant \frac{8}{\rho}\Big(\frac{\mu^2}{\rho} - \mu\theta^2\Big)|\xi|^2 + \frac{8}{\rho}\Big(\frac{2\mu}{\rho} - \theta^2\Big)\sum_{j,k=1}^{n} r_{jk}\xi_k\xi_k$$

$$- \frac{8}{\rho^2}\sum_{k=1}^{n}\|\partial_k R\||\xi||x-\beta|\sum_{m=1}^{n}|r_{km} + \mu\delta_{km}||\xi_m|$$

$$- \frac{4}{\rho^2}\sum_{j,k=1}^{n}\|\partial_k R\||\xi|^2|r_{jk} + \mu\delta_{jk}||(x-\beta)_j|,$$

where we used the relation

$$\Big|\sum_{j,k=1}^{n} r_{jk}\xi_j\eta_k\Big| \leqslant \|R\||\xi||\eta|$$

which follows from the Cauchy–Schwartz inequality. Using this inequality again, we conclude that

$$\mathcal{H} \geqslant \frac{8}{\rho}\Big(\frac{\mu^2}{\rho} - \mu\theta^2\Big)|\xi|^2 - \frac{12}{\rho^2}\|R + \mu I\|\|\nabla R\||x-\beta||\xi|^2.$$

Hence the positivity of $\mathcal{H}$ follows from (2.1).

Since $|x-\beta|^2 > \theta^2 t^2$, we have

$$A(x, \nabla\psi(x)) = 4\theta^4 t^2 - \frac{\mu}{\rho}4|x-\beta|^2 - \sum_{j,k=1}^{n} \frac{r_{jk}}{\rho}4(x-\beta)_j(x-\beta)_k$$
$$< 4((\theta^2 - \frac{\mu}{\rho})|x-\beta|^2 - \sum_{j,k=1}^{n} \frac{r_{jk}}{\rho}(x-\beta)_j(x-\beta)_k) < 0$$

on $\overline{\Omega} \cap \{|x-\beta|^2 > \theta^2 t^2\}$ by the condition (2.2) and the definition of $\Omega_0$. So, $\nabla\psi$ is not characteristic on this set. □

# 3 Proof of Carleman Estimates for Scalar Operators

In the following, $\zeta(\varphi)(x) = \xi + i\tau\nabla\varphi(x)$. We introduce the differential quadratic form

$$\mathcal{F}(x,\tau,D,\overline{D})v\overline{v} = |A(x, D + i\tau\nabla\varphi(x))v|^2 - |A(x, D - i\tau\nabla\varphi(x))v|^2. \quad (3.1)$$

This quadratic form is of order (3, 2) since the coefficients of the principal part of $A$ are real valued. By [6, Lemma 8.2.2], there exists a differential quadratic form $\mathcal{G}(x,\tau,D,\overline{D})$ of order (2, 1) such that

$$\int_{\Omega} \mathcal{G}(x,D,\overline{D})v\overline{v} = \int_{\Omega} \mathcal{F}(x,D,\overline{D})v\overline{v} \quad (3.2)$$

with the symbol

$$\mathcal{G}(x,\tau,\xi,\xi) = \frac{1}{2}\sum \frac{\partial^2}{\partial x_k \partial \eta_k}\mathcal{F}(x,\tau,\zeta,\overline{\zeta}), \quad \zeta = \xi + i\eta, \quad at \;\; \eta = 0,$$

where

$$\mathcal{F}(x,\tau,\zeta,\overline{\zeta}) = A(x,\zeta + i\tau\nabla\varphi)A(x,\overline{\zeta} - i\tau\nabla\varphi) - A(x,\zeta - i\tau\nabla\varphi)A(x,\overline{\zeta} + i\tau\nabla\varphi).$$

**Lemma 3.1.** *We have*

$$\mathcal{G}(x,\tau,\xi,\xi) = 2\tau\sum \frac{\partial A}{\partial \zeta_j}\overline{\frac{\partial A}{\partial \zeta_k}}\partial_j\partial_k\varphi + 2\Im\sum \partial_k A\overline{\frac{\partial A}{\partial \zeta_k}}$$

$$+\,2\Im \sum A\Big(\overline{\frac{\partial^2 A}{\partial\zeta_k\partial x_k}} - i\tau\, \overline{\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k}}\partial_j\partial_k\varphi\Big), \tag{3.3}$$

*where* $A, \partial_k A, \ldots$ *are taken at* $(x, \zeta(\varphi)(x))$.

*Proof.* Indeed, at $\eta = 0$

$$\frac{1}{2}\sum \frac{\partial^2}{\partial x_k\partial\eta_k}\mathcal{F}(x, \xi + i\eta, \xi - i\eta)$$
$$= \frac{1}{2}\sum \partial_k\Big(i\frac{\partial A}{\partial\zeta_k}(x, \xi + i\tau\nabla\varphi)A(x, \xi - i\tau\nabla\varphi)$$
$$- iA(x, \xi + i\tau\nabla\varphi)\frac{\partial A}{\partial\zeta_k}(x, \xi - i\tau\nabla\varphi)$$
$$- i\frac{\partial A}{\partial\zeta_k}(x, \xi - i\tau\nabla\varphi)A(x, \xi + i\tau\nabla\varphi)$$
$$+ iA(x, \xi - i\tau\nabla\varphi)\frac{\partial A}{\partial\zeta_k}(x, \xi + i\tau\nabla\varphi)\Big)$$
$$= i\sum \partial_k\Big(\frac{\partial A}{\partial\zeta_k}(x, \zeta(\varphi))A(x, \overline{\zeta}(\varphi)) - \frac{\partial A}{\partial\zeta_k}(x, \overline{\zeta}(\varphi))A(x, \zeta(\varphi))\Big).$$

Using the fact that $i(z\overline{w} - \overline{z}w) = -2\Im(z\overline{w})$, we yield

$$\mathcal{G}(x, \tau, \xi, \xi) = -2\Im\sum \partial_k\Big(\frac{\partial A}{\partial\zeta_k}(x, \zeta(\varphi)(x))A(x, \overline{\zeta}(\varphi(x))\Big)$$
$$= -2\Im\sum\Big(\Big(\frac{\partial^2 A}{\partial x_k\partial\zeta_k}(x, \zeta(\varphi)) + i\tau\partial_j\partial_k\varphi\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k}(x, \zeta(\varphi))\Big)A(x, \overline{\zeta}(\varphi))$$
$$+ \frac{\partial A}{\partial\zeta_k}(x, \zeta(\varphi))\frac{\partial A}{\partial x_k}(x, \overline{\zeta}(\varphi)) - i\tau\partial_j\partial_k\varphi\frac{\partial A}{\partial\zeta_k}(x, \zeta(\varphi))\frac{\partial A}{\partial\zeta_j}(x, \overline{\zeta}(\varphi))\Big) \tag{3.4}$$

by the chain rule and $\dfrac{\partial\zeta_j}{\partial x_k} = i\tau\partial_j\partial_k\varphi$. Observing that, in the notation of Lemma 3.1, $A(, \overline{\zeta}(\varphi)) = \overline{A}$, $-\Im(z\overline{w}) = \Im(\overline{z}w)$ and reminding that the coefficients of $A$ are real-valued and, consequently, $\sum\partial_j\partial_k\varphi\dfrac{\partial A}{\partial\zeta_k}\overline{\dfrac{\partial A}{\partial\zeta_j}}$ is real-valued, we obtain (3.3) from (3.4). □

The following differentiation formulas are obtained from (1.4) and are used in the proofs below:

$$\partial_j\varphi = \gamma\varphi\partial_j\psi\,,\ \partial_j\partial_k\varphi = \gamma\varphi\partial_j\partial_k\psi + \gamma^2\varphi\partial_j\psi\partial_k\psi. \tag{3.5}$$

By these formulas and Lemma 3.1, a standard calculation yields

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) = \mathcal{G}_1(x,\tau,\xi,\xi) + \mathcal{G}_2(x,\tau,\xi,\xi) + \mathcal{G}_3(x,\tau,\xi,\xi) + \mathcal{G}_4(x,\tau,\xi,\xi), \tag{3.6}$$

where

$$\mathcal{G}_1(x,\tau,\xi,\xi) = 8\gamma\varphi\sum a^{jm}a^{kl}(\xi_m\xi_l + \sigma^2\partial_m\psi\partial_l\psi)\partial_j\partial_k\psi,$$

$$\mathcal{G}_2(x,\tau,\xi,\xi) = 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(\sigma^2\partial_j\psi\partial_m\psi\partial_l\psi + 2\xi_m\xi_l\partial_j\psi - \xi_j\xi_m\partial_l\psi),$$

$$\begin{aligned}\mathcal{G}_3(x,\tau,\xi,\xi) = 4\gamma\varphi(2&\sum a^{km}\partial_j a^{lj}\partial_k\psi\xi_l\xi_m \\ &- \sum a^{jk}(\partial_m a^{lm}\partial_l\psi + a^{lm}\partial_l\partial_m\psi)(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi)),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_4(x,\tau,\xi,\xi) = 4\gamma^2\varphi\Big(&\Big(2\sum a^{jm}\xi_m\partial_j\psi\Big)^2 + 2\sigma^2\Big(\sum a^{jm}\partial_j\psi\partial_m\psi\Big)^2 \\ &- \Big(\sum a^{lm}(\xi_l\xi_m - \sigma^2\partial_l\psi\partial_m\psi)\Big)\Big(\sum a^{jk}\partial_j\psi\partial_k\psi\Big)\Big).\end{aligned}$$

Note that the terms of $\tau^{-1}\mathcal{G}$ with highest powers of $\lambda$ are collected in $\mathcal{G}_4$.

*Proof of Theorem* 1.1. First, make the substitution $u = e^{-\tau\varphi}v$. It is obvious that $D_k(e^{-\tau\varphi}v) = e^{-\tau\varphi}(D_k + i\tau\partial_k\varphi)v$. Hence

$$\sum a^{jk}D_jD_k(e^{-\tau\varphi}v) = \sum a^{jk}e^{-\tau\varphi}(D_j + i\tau\partial_j\varphi)(D_k + i\tau\partial_k\varphi)v.$$

Accordingly, the bound (1.5) is transformed into

$$\sum\int\limits_{\Omega}\sigma^{3-2|\alpha|}|\partial^\alpha v|^2 \leqslant C\int\limits_{\Omega}|A(,D + i\tau\nabla\varphi)v|^2. \tag{3.7}$$

**Lemma 3.2.** *Under the assumptions of Theorem* 1.1*, for any* $\varepsilon_0$ *there is* $C$ *such that*

$$\gamma\varphi(x)(2K - \varepsilon_0)|\zeta(\varphi)(x))|^2 \leqslant \tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) + \gamma\varphi(x)C\gamma^2\frac{|A(x,\zeta(\varphi)(x))|^2}{|\zeta(\varphi)(x)|^2} \tag{3.8}$$

*for all* $C < \gamma$*,* $\xi\in\mathbb{R}^n$*, and* $x\in\overline{\Omega}$*.*

*Proof.* By homogeneity, we can assume that $|\zeta(\varphi)|(x) = 1$. In the proof, we use the relation

$$\begin{aligned}A(x,\zeta(\varphi)(x)) &= \sum_{j,k=1}^{n}a^{jk}(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi) + 2i\sum_{j,k=1}^{n}a^{jk}\sigma\xi_j\partial_k\psi \\ &= A(x,\xi) - \sigma^2 A(x,\nabla\psi(x)) + 2i\sigma\sum\frac{\partial A}{\partial\zeta_j}(x,\xi)\partial_j\psi(x).\end{aligned} \tag{3.9}$$

To show (3.8), we use the pseudoconvexity of $\psi$ and consider four cases.

*Case* 1.

$$\sigma = 0, \quad A(x,\xi) = 0, \quad \sum \frac{\partial A}{\partial \zeta_j}(x,\xi)\partial_j\psi(x) = 0. \tag{3.10}$$

Then

$$\sigma = 0, \quad \sum a^{jk}\xi_j\xi_k = 0, \quad \sum a^{jk}\xi_j\partial_k\psi = 0,$$

and from (3.6) we find

$$\begin{aligned}
&\tau^{-1}\mathcal{G}(x,0,\xi,\xi) \\
&= 2\gamma\varphi\sum \partial_j\partial_k\psi 2a^{jm}\xi_m 2a^{kl}\xi_l + 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi) \\
&= 2\gamma\varphi\sum \partial_j\partial_k\psi\frac{\partial A}{\partial\zeta_j}\frac{\partial A}{\partial\zeta_k} + 2\gamma\varphi\sum\Big(\Big(\partial_k\frac{\partial A}{\partial\zeta_j}\Big)\frac{\partial A}{\partial\zeta_k} - (\partial_k A)\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k}\Big)\partial_j\psi(x,\xi) \\
&\geqslant 2\gamma\varphi K
\end{aligned}$$

by the pseudoconvexity of $\psi$ (1.2).

*Case* 2.

$$\sigma < \delta, \quad |\gamma(A(x,\xi) - \sigma^2 A(x,\nabla\psi(x)))| < \delta, \tag{3.11}$$

where $\delta$ is a (small) positive number to be chosen later.

Using (3.6) as in Case 1, bounding the terms with $\sigma^2$ by $-C\gamma\varphi\delta$, and dropping the second (positive) term in $\mathcal{G}_4$, we obtain

$$\begin{aligned}
\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) &\geqslant 2\gamma\varphi\sum 2a^{jm}\xi_m 2a^{kl}\xi_l\partial_j\partial_k\psi - C\gamma\varphi\delta^2 \\
&\quad + 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi) + 4\gamma\varphi 2\sum \partial_j a^{lj}\xi_l(a^{km}\partial_k\psi\xi_m) \\
&\quad - 4\gamma\varphi\sum a^{jk}(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi)\sum(\partial_m a^{lm}\partial_l\psi + a^{lm}\partial_l\partial_m\psi) \\
&\quad + 8\gamma^2\varphi\Big(\sum a^{jm}\xi_m\partial_j\psi\Big)^2 \\
&\quad - \gamma\varphi\Big(\gamma\sum a^{jk}(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi)\Big)\Big(\sum a^{lm}\partial_l\psi\partial_m\psi\Big) \\
&\geqslant 2\gamma\varphi\Big(\sum\partial_j\partial_k\psi 2a^{jm}\xi_m 2a^{kl}\xi_l + 2\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi)\Big) \\
&\qquad - C\gamma\varphi\delta + 8\gamma\varphi\gamma\Big(\sum a^{jk}\xi_j\partial_k\psi\Big)^2,
\end{aligned} \tag{3.12}$$

due to (3.9) and (3.11).

In addition to (3.11), we assume that

$$\Big|\sum \frac{\partial A}{\partial \zeta_j}(x,\xi)\partial_j\psi(x)\Big| < \delta. \tag{3.13}$$

Then

$$\begin{aligned} h(x,\xi,\sigma) &= \sum \partial_j\partial_k\psi(x)2a^{jm}(x)\xi_m 2a^{kl}(x)\xi_l \\ &\quad + 2\sum a^{lk}\partial_k a^{jm}(x)(2\xi_l\xi_m\partial_j\psi(x) - \xi_j\xi_m\partial_l\psi(x)) \\ &\geqslant K - \varepsilon(\delta) \end{aligned} \tag{3.14}$$

by the continuity arguments, compactness of the set

$$M = \{(x,\xi,\sigma) : x \in \overline{\Omega}, |\xi|^2 + \sigma^2|\nabla\psi(x)|^2 = 1\},$$

and (3.11). Here, $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

Indeed, assuming the opposite of (3.14), we yield a positive number $\varepsilon_1$ and a sequence $(x(k),\xi(k),\sigma(k)) \in M$ such that $h((x(k),\xi(k),\sigma(k)) \leqslant K - \varepsilon_1$ and (3.11), (3.13) hold with $\delta = k^{-1}$. Since $M$ is compact (by extracting a subsequence, if necessary), we may assume that $(x(k),\xi(k),\sigma(k)) \to (x,\xi,0) \in M$ as $k \to +\infty$. By continuity, $h(x,\xi,0) \leqslant K - \varepsilon_1$. On the other hand, by the choice of the sequence, using that $1 \leqslant \gamma$, we find that $(x,\xi,0)$ satisfies (3.10). Hence, by case 1, $h(x,\xi,0) \geqslant K$ and we obtain a contradiction.

Because of (3.14), the right-hand side of (3.12) is greater than

$$\gamma\varphi(2K - \varepsilon(\delta) - C\delta) \geqslant \gamma\varphi(2K - \varepsilon_0).$$

Here, we let $\delta < \frac{1}{C}$, so that $\varepsilon(\delta) + C\delta < \varepsilon_0$. From now on, we fix such $\delta$ and denote it by $\delta_0$. We can choose $\delta_0$ to be dependent on the same parameters as $C$.

If

$$\Big|\sum \frac{\partial A}{\partial \zeta_j}(x,\xi)\partial_j\psi(x)\Big| \geqslant \delta_0,$$

then, using (3.11) with $\delta = \delta_0$, we conclude that the right-hand side of (3.12) is greater than

$$-C\gamma\varphi + 8\gamma\varphi\gamma\delta_0^2 \geqslant \gamma\varphi 2K$$

when $\gamma > 8^{-1}\delta_0^{-2}(C + 2K)$.

Finally, the condition (3.11) with $\delta = \delta_0$ implies (3.8).

To conclude the proof, we observe that, by (3.9), in addition to (3.11) only the following cases 3 and 4 are possible.

*Case* 3. $\sigma > \delta_0$, $|\gamma\Re A(x,\zeta(\varphi)(x))| < \delta_0$.

Using (3.6), as above we yield

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) \geqslant -C\gamma\varphi(x) + \mathcal{G}_4(x,\tau,\xi,\xi)$$

[by dropping the first (positive) term in $\mathcal{G}_4$, using that $\nabla\psi$ is non-characteristic, and bounding the last term in $\mathcal{G}_4$ from (3.9) and the $\delta_0$-smallness of $|\gamma\Re A|$]

$$\geqslant -C\gamma\varphi(x) + 8C^{-1}\gamma^2\varphi\delta_0^2 \geqslant 2\gamma\varphi(x)K,$$

when we choose $\gamma > C^2$.

*Case* 4. $|\gamma A(x,\zeta(\varphi)(x)| > \delta_0$.
From (3.6) we similarly have

$$\begin{aligned}
&\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) + \gamma\varphi(x)C_1|\gamma A(x,\zeta(\varphi)(x))|^2\\
&\geqslant -C\gamma\varphi(x) - C\gamma^2\varphi|A(x,\zeta(\varphi)(x))| + \gamma\varphi C_1|\gamma A(x,\zeta(\varphi)(x))|^2\\
&\geqslant -C\gamma\varphi(x) - C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))| + \gamma\varphi C_1|\gamma A(x,\zeta(\varphi)(x))|^2\\
&\geqslant -C\gamma\varphi(x) + C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))|\Big(\frac{C_1}{2C}|\gamma A(x,\zeta(\varphi)(x))| - 1\Big)\\
&\quad + \gamma\varphi(x)\frac{C_1}{2}|\gamma A(x,\zeta(\varphi)(x))|^2\\
&\geqslant -C\gamma\varphi(x) + C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))|\Big(\frac{C_1\delta_0}{2C} - 1\Big) + \gamma\varphi(x)\frac{C_1}{2}\delta_0^2\\
&\geqslant K\gamma\varphi(x)
\end{aligned}$$

if $C_1 > \dfrac{2C}{\delta_0} + \dfrac{C+2K}{2\delta_0^2}$. □

We fix $x_0 \in \overline{\Omega}$, introduce the norm

$$|||v|||_{-1} = \Big(\int \frac{|\widehat{v}(\xi)|^2}{|\xi|^2 + \tau^2\gamma^2\varphi^2(x_0)|\nabla\psi(x_0)|^2}d\xi\Big)^{\frac{1}{2}}, \tag{3.15}$$

and observe that

$$|||v|||_{-1} \leqslant C\tau^{-1}||v||_2. \tag{3.16}$$

**Lemma 3.3.** *There is a function $\varepsilon(\delta;\gamma) \to 0$, as $\delta \to 0$ and $\gamma$ is fixed, and $C(\gamma)$ such that*

$$\tau^{-1}|(\mathcal{G}(x_0,\tau,D,\overline{D}) - \mathcal{G}(,\tau,D,\overline{D}))v\overline{v}| \leqslant \varepsilon(\delta;\gamma)\sum_{|\alpha|\leqslant 1}\tau^{2-2|\alpha|}|\partial^\alpha v|^2, \tag{3.17}$$

$$\begin{aligned}
&|||A(x_0, D + i\tau\nabla\varphi(x_0))v - A(, D + i\tau\nabla\varphi)v|||_{-1}^2\\
&\quad\leqslant (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1})\sum_{|\alpha|\leqslant 1}(\gamma\tau\varphi(x_0))^{2-2|\alpha|}\int|\partial^\alpha v|^2
\end{aligned} \tag{3.18}$$

*for all* $v \in C_0^2(B(x_0;\delta))$.

*Proof.* By (3.6), we have

$$\begin{aligned}
&\tau^{-1}(\mathcal{G}(x_0,\tau,D,\overline{D}) - \mathcal{G}(,\tau,D,\overline{D}))v\overline{v} \\
&\quad = \sum(\gamma(\varphi(x_0)a_1^{jk}(x_0) - \varphi(x)a_1^{jk}(x))\partial_j v(x)\partial_k v(x)) \\
&\quad + \gamma^3\tau^2(\varphi(x_0)^2 a_2^{jk}(x_0) - \varphi(x_0)^2 a_2^{jk})v(x)v(x)
\end{aligned}$$

$$\begin{aligned}
&+ \gamma^2\sum((\varphi(x_0)a_3^{jk}(x_0) - \varphi(x)a_3^{jk}(x))\partial_j v(x)\partial_k v(x)) \\
&+ \gamma^4\tau^2(\varphi(x_0)^2 a_4^{jk}(x_0) - \varphi(x_0)^2 a_4^{jk}(x))v(x)v(x),
\end{aligned}$$

where $a_l^{jk}$ are continuous functions determined only by $A$ and $\psi$. Since $|\varphi(x_0)^m a_l^{jk}(x_0) - \varphi(x)^m a_l^{jk}(x)| \leqslant \varepsilon(\delta;\gamma)$ for $|x - x_0| < \delta$, (3.17) follows by the triangle inequality.

We have

$$\begin{aligned}
&|||A(x_0, D + i\tau\nabla\varphi(x_0))v - A(, D + i\tau\nabla\varphi)v|||_{-1} \\
&\leqslant \Big|\Big|\Big|\sum(a^{jk}(x_0) - a^{jk})(\partial_j - \tau\partial_j\varphi(x_0))(\partial_k - \tau\partial_k\varphi(x_0))v\Big|\Big|\Big|_{-1} \\
&\leqslant C(\delta + \tau^{-1})||(\partial - \tau\partial\varphi)v||_2 \\
&\leqslant (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1})\sum_{|\alpha|\leqslant 1}\tau^{1-|\alpha|}||\partial^\alpha v||^2 \qquad (3.19)
\end{aligned}$$

by [6, Lemma 8.4.1].

Furthermore,

$$\begin{aligned}
&|||A(, D+i\tau\nabla\varphi(x_0))v - A(, D + i\tau\nabla\varphi)v|||_{-1} \\
&\quad = \Big|\Big|\Big|\sum a^{jk}((\partial_j - \tau\partial_j\varphi(x_0))(\partial_k - \tau\partial_k\varphi(x_0)) \\
&\qquad\qquad - (\partial_j - \tau\partial_j\varphi)(\partial_k - \tau\partial_k\varphi))v\Big|\Big|\Big|_{-1} \\
&\quad \leqslant \Big|\Big|\Big|\sum a^{jk}(\tau^2(\partial_j\varphi(x_0)\partial_k\varphi_k(x_0) - \partial_j\varphi\partial_k\varphi) \\
&\qquad\qquad + 2\tau(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k + \tau(\partial_j\partial_k\varphi)v)\Big|\Big|\Big|_{-1} \\
&\quad \leqslant \sum\tau^2|||a^{jk}(\partial_j\varphi\partial_k\varphi - \partial_j\varphi(x_0)\partial_k\varphi(x_0))v|||_{-1} \\
&\quad + 2\sum\tau|||(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k v|||_{-1} + \tau\sum|||\partial_j\partial_k\varphi v|||_{-1}.
\end{aligned}$$

Using the property of the norm (3.15), we find

$$
\begin{aligned}
&|||A(,D+i\tau\nabla\varphi)v - A(,D+i\tau\nabla\varphi(x_0))v|||_{-1}\\
&\leqslant C\tau\sum ||a^{jk}(\partial_j\varphi\partial_k\varphi - \partial_j\varphi(x_0)\partial_k\varphi(x_0))v||_2\\
&\qquad + 2\sum ||(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k v||_2 + \sum ||\partial_j\partial_k\varphi v||_2\\
&\leqslant \tau\varepsilon(\delta;\gamma)||v||_2 + \varepsilon(\delta;\gamma)\sum ||\partial_k v||_2 + C(\gamma)||v||_2
\end{aligned}
$$

$$
\leqslant (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1}) \sum_{|\alpha|\leqslant 1} \tau^{1-|\alpha|}||\partial^\alpha v||_2
$$

Using (3.19), we have

$$
\begin{aligned}
&|||A(x_0,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x))v|||_{-1}\\
&\leqslant |||A(x_0,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x_0))v|||_{-1}\\
&\qquad + |||A(x,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x))v|||_{-1}\\
&\leqslant (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1}) \sum_{|\alpha|\leqslant 1} \tau^{1-|\alpha|}||\partial^\alpha v||_2.
\end{aligned}
$$

The proof of the lemma is complete. □

Now, we continue the proof of Theorem 1.1. By the Parseval identity,

$$
(\tau^2|\nabla\varphi(x_0)|^2)^{m-|\alpha|}\int |\partial^\alpha v|^2 dx \leqslant (2\pi)^{-n}\int |\zeta|^{2m}(\varphi)(x_0)|\widehat{v}(\xi)|^2 d\xi.
$$

Hence, multiplying the inequality (3.8) by $|\widehat{v}(\xi)|^2$, $v \in C_0^2(\Omega_\varepsilon)$, and integrating over $\mathbf{R}^n$, we yield

$$
\begin{aligned}
&C^{-1}\gamma\varphi(x_0)\sum_{|\alpha|\leqslant 1}\int (\gamma\tau\varphi(x_0))^{2-2|\alpha|}|\partial^\alpha v|^2\\
&\leqslant \tau^{-1}\int \mathcal{G}(x_0,\tau,D,\overline{D})v\overline{v} + \gamma\varphi(x_0)\gamma^2\int \frac{|A(x_0,\zeta(\varphi)(x_0))|^2}{|\zeta(\varphi)(x_0)|^2}|\widehat{v}(\xi)|^2 d\xi\\
&\leqslant \tau^{-1}\int \mathcal{G}(x_0,\tau,D,\overline{D})v\overline{v} + \gamma\varphi(x_0)\gamma^2|||A(x_0,D+i\tau\nabla\varphi(x_0))v|||_{-1}^2\\
&\leqslant \tau^{-1}\int \mathcal{G}(x,\tau,D,\overline{D})v\overline{v} + \varepsilon(\delta;\gamma)\sum_{|\alpha|\leqslant 1}\tau^{2-2|\alpha|}\int |\partial^\alpha v|^2\\
&\quad + \gamma\varphi(x_0)\gamma^2|||A(,D+i\tau\nabla\varphi)v|||_{-1}^2\\
&\quad + (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-2})\sum_{|\alpha|\leqslant 1}\tau^{3-2|\alpha|}\int |\partial^\alpha v|^2
\end{aligned}
\tag{3.20}
$$

for $v \in C_0^2(\Omega_\varepsilon \cap B(x_0, \delta))$. Here, we used Lemma 3.3 and the elementary inequality $a^2 \leqslant 2b^2 + 2(b-a)^2$. Choosing $\delta > 0$ small and $\tau$ large enough so that $(2C)^{-1}\gamma\varphi(x_0)(\gamma\tau\varphi(x_0))^{2-2|\alpha|} > (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-2})\tau^{2-2|\alpha|}$, we absorb the second and fourth terms on the right-hand side of the inequality (3.20) to arrive at the inequality

$$\sum_{|\alpha|\leqslant 1} \int (\gamma\tau\varphi(x_0))^{3-2|\alpha|}|\partial^\alpha v|^2$$

$$\leqslant C(\int \mathcal{G}(,\tau, D, \overline{D})v\overline{v} + \tau\gamma\varphi(x_0)\gamma^2|||A(, D + i\tau\nabla\varphi)v|||_{-1}^2).$$

As above, by choosing large $\tau > C(\gamma)$, one can replace $\varphi(x_0)$ on the left-hand side of this inequality by $\varphi$. Using (3.1), (3.2), and the property (3.16) of the norm $|||\,|||_{-1}$, we conclude that

$$\sum_{|\alpha|\leqslant 1} \int (\gamma\tau\varphi)^{3-2|\alpha|}|\partial^\alpha v|^2$$

$$\leqslant C||A(, D + i\tau\nabla\varphi)v||_2^2 + C(\gamma)\tau^{-1}||A(, D + i\tau\nabla\varphi)v||_2^2$$

for $v \in C_0^2(B(x_0;\delta))$. Choosing $\tau > C(\gamma)$, we eliminate the second term on the right-hand side. Now, the bound (3.7) follows by the partition of unity argument. Since our choice of $\delta_0$ depends on $\gamma$, we give this argument in some detail.

The balls $B(x_0;\delta_0)$ form an open covering of the compact set $\overline{\Omega}$. Hence we can find a finite subcovering $B(x_{0j};\delta_0)$ and a special partition of unity $\chi_j(;\gamma)$ subordinated to this subcovering. In particular, $\chi_j \in C_0^2(B(x_{0j};\delta_0))$, $0 \leqslant \chi_j \leqslant 1$, and $\sum \chi_j^2 = 1$ on $\overline{\Omega}$. By the Leibniz formula,

$$\partial^\alpha(\chi_j v) = \chi_j \partial^\alpha v + (\partial^\alpha \chi_j)v,$$

$$A(, D + i\tau\nabla\varphi)(\chi_j v) = \chi_j A(, D + i\tau\nabla\varphi)v) + \sum_{|\beta|\leqslant 1} a^\beta \tau^{1-|\beta|}\partial^\beta v$$

with $|a^\beta| \leqslant C(\gamma)$. Hence, applying the Carleman estimate (3.7) to $\chi_j v$ and using the elementary inequality $|a+b|^2 \geqslant \frac{1}{2}a^2 - b^2$, we obtain

$$\frac{1}{2}\sum_{|\alpha|\leqslant 1}\int \sigma^{3-2|\alpha|}|\chi_j\partial^\alpha v|^2 - \sum_{|\alpha|=1}\int \sigma^3|(\partial^\alpha(\chi_j))v)|^2$$

$$\leqslant C||\chi_j A(, D + i\tau\nabla\varphi)v||_2^2 + C(\gamma)\sum_{|\beta|\leqslant 1}\tau^{2-2|\beta|}||\partial^\beta v||_2^2.$$

Summing up over $j = 1, \ldots, J$ and using that $\sum \chi_j^2 = 1$, we yield

$$\frac{1}{2}\sum_{|\alpha|\leqslant 1}\int \sigma^{3-2|\alpha|}|\partial^\alpha v|^2 - \sum_{|\alpha|=1,j\leqslant J}\int \sigma|(\partial^\alpha(\chi_j))v)|^2$$

$$\leqslant C||A(,D+i\tau\nabla\varphi)v||_2^2 + C(\gamma)\sum_{|\beta|\leqslant 1}\tau^{2-2|\beta|}||\partial^\beta v||_2^2.$$

Since the highest powers of $\tau$ are in the first term on the left-hand side, choosing $C(\gamma) < \tau$, we absorb by this term the second term on the left-hand side and the second term on the right-hand side. □

## 4 Proof of Carleman Estimates for Elasticity System

**Lemma 4.1.** *Let $|\nabla\psi| > 0$ on $\overline{\Omega}$. Then for a second order elliptic operator $A$ there are constants $C$ and $C_0(\gamma)$ such that*

$$\gamma\int_\Omega \sigma^{4-2|\alpha|}e^{2\tau\varphi}|\partial^\alpha v|^2 \leqslant C\int_\Omega \sigma e^{2\tau\varphi}|Av|^2 \tag{4.1}$$

*for all $v \in C_0^2(\Omega)$, $|\alpha| \leqslant 2$, $C < \gamma$, and $C_0(\gamma) < \tau$.*

*Proof.* We apply the Carleman estimate in [4]

$$\sum_{|\alpha|\leqslant 2}\sqrt{\gamma}\,||\sigma^{\frac{3}{2}-|\alpha|}e^{\tau\varphi}\partial^\alpha u|| \leqslant C\,||e^{\tau\varphi}A(x,D)u|| \tag{4.2}$$

to $u = \sigma^{\frac{1}{2}}v$. By the Leibniz formula,

$$\partial^\alpha(\sigma^{\frac{1}{2}}v) = \sigma^{\frac{1}{2}}\partial^\alpha v + \tau^{\frac{1}{2}}A_{|\alpha|-1}(x,D)v, \quad |\alpha| = 1,2,$$

$$A(x,D)(\sigma^{\frac{1}{2}}v) = \sigma^{\frac{1}{2}}A(x,D)v + \tau^{\frac{1}{2}}A_1(x,D)v,$$

where $A_m$ is a linear partial differential operator of order $m$ with coefficients bounded by $C(\gamma)$. Using these relations with $|\alpha| = 1$ and the triangle inequality, from (4.2), we get

$$\sqrt{\gamma}\,||\sigma e^{\tau\varphi}\nabla v|| - C(\gamma)||\tau e^{\tau\varphi}v|| \leqslant C\,||\sigma^{\frac{1}{2}}e^{\tau\varphi}A(x,D)v|| + C(\gamma)\sum_{|\alpha|\leqslant 1}||\tau^{\frac{1}{2}}e^{\tau\varphi}\partial^\alpha v||.$$

Similarly, when $|\alpha| = 2$,

$$\sqrt{\gamma}\,||e^{\tau\varphi}\partial^\alpha v|| - C(\gamma)\sum_{|\alpha|\leqslant 1}||e^{\tau\varphi}\partial^\alpha v||$$

$$\leqslant C\,||\sigma^{\frac{1}{2}}e^{\tau\varphi}A(x,D)v|| + C(\gamma)\sum_{|\alpha|\leqslant 1}||\tau^{\frac{1}{2}}e^{\tau\varphi}\partial^\alpha v||.$$

Summing the inequalities over $|\alpha| \leqslant 2$, we yield

$$\sqrt{\gamma}\sum_{|\alpha|\leqslant 2}||\sigma^{2-|\alpha|}e^{\tau\varphi}\partial^\alpha v|| - C(\gamma)\sum_{|\alpha|\leqslant 1}\tau^{1-|\alpha|}||e^{\tau\varphi}\partial^\alpha v||$$

$$\leqslant C\,||\sigma^{\frac{1}{2}}e^{\tau\varphi}Av|| + C(\gamma)||\tau^{\frac{1}{2}}e^{\tau\varphi}\partial^\alpha v||.$$

Since $\sigma = \tau\gamma\varphi$, $1 \leqslant \gamma$, $1 \leqslant \varphi$, the second terms on the left-hand side and on the right-hand side are absorbed by the first term on the left-hand side by choosing $\tau > C(\gamma)$. □

*Proof of Theorem* 1.2. We want to apply Carleman estimates to the system $\mathbf{A}_R\mathbf{u} = \mathbf{f}$. Unfortunately, we have no Carleman estimates for systems. To use Carleman estimates for scalar equations, we extend this system to a new principally triangular system. By the standard substitution ($\mathbf{u}, v = \operatorname{div}\mathbf{u}$, $\mathbf{w} = \operatorname{curl}\mathbf{u}$), the system (1.6) can be reduced [11, Proposition 2.1] to a new system, where the leading part is a special lower triangular matrix differential operator with the wave operators on the diagonal:

$$\Box(\mu;R)\mathbf{u} = \frac{\mathbf{f}}{\rho} + A_{1;1}(\mathbf{u},v),$$

$$\Box(\lambda+2\mu;R)v = \operatorname{div}\frac{\mathbf{f}}{\rho} + \sum_{jk}\nabla\Big(\frac{r_{jk}}{\rho}\Big)\cdot\partial_j\partial_k\mathbf{u} + A_{2;1}(\mathbf{u},v,\mathbf{w}), \tag{4.3}$$

$$\Box(\mu;R)\mathbf{w} = \operatorname{curl}\frac{\mathbf{f}}{\rho} + \sum_{jk}\nabla\Big(\frac{r_{jk}}{\rho}\Big)\times\partial_j\partial_k\mathbf{u} + A_{3;1}(\mathbf{u},v,\mathbf{w}),$$

where $A_{j;1}$ are first order differential operators.

Applying Theorem 1.1 to each of seven scalar differential operators forming the extended system (4.3) and summing up seven Carleman estimates, we get

$$\int_\Omega(\sigma|\nabla_{x,t}\mathbf{u}|^2 + \sigma|\nabla_{x,t}v|^2 + \sigma|\nabla_{x,t}\mathbf{w}|^2 + \sigma^3|\mathbf{u}|^2 + \sigma^3|v|^2 + \sigma^3|\mathbf{w}|^2)e^{2\tau\varphi}$$

$$\leqslant C\int_\Omega(|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} + C\int_\Omega\sum_{j,k=1}^{3}|\partial_j\partial_k\mathbf{u}|^2e^{2\tau\varphi}$$

$$+C \int_{\Omega} (|\nabla \mathbf{u}|^2 + |\nabla u|^2 + |\nabla \mathbf{w}|^2 + |\mathbf{u}|^2 + v^2 + |\mathbf{w}|^2) e^{2\tau\varphi}.$$

Taking $\tau > 2C$, we can absorb the third integral on the right-hand side by the left-hand side arriving at the inequality

$$\int_{\Omega} (\sigma|\nabla_{x,t}\mathbf{u}|^2 + \sigma|\nabla_{x,t}v|^2 + \sigma|\nabla_{x,t}\mathbf{w}|^2 + \sigma^3|\mathbf{u}|^2 + \sigma^3|v|^2 + \sigma^3|\mathbf{w}|^2) e^{2\tau\varphi}$$

$$\leqslant C \int_{\Omega} (|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2) e^{2\tau\varphi} + C \int_{\Omega} \sum_{j,k=1}^{3} |\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi}. \qquad (4.4)$$

To eliminate the second order derivatives on the right-hand side, we need the second large parameter $\gamma$. By Lemma 4.1,

$$\begin{aligned}
\gamma \int_{\Omega} \sum_{j,k=1}^{3} |\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi} &\leqslant C \int_{\Omega} \sigma|\Delta\mathbf{u}|^2 e^{2\tau\varphi} \\
&\leqslant C \int_{\Omega} \sigma(|\nabla v|^2 + |\nabla\mathbf{w}|^2) e^{2\tau\varphi} \\
&\leqslant C \int_{\Omega} (|\mathbf{f}|^2 + |\nabla\mathbf{f}|^2) e^{2\tau\varphi} + C \int_{\Omega} |\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi},
\end{aligned}$$

where we used the known identity $\Delta\mathbf{u} = \nabla v - \operatorname{curl} \mathbf{w}$ and (4.4). Choosing $\gamma > 2C$, we see that the second order derivatives term on the right-hand side is absorbed by the left-hand side. This yields

$$\gamma \int_{\Omega} \sum_{j,k=1}^{3} |\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi} \leqslant C \int_{\Omega} (|\mathbf{f}|^2 + |\nabla\mathbf{f}|^2) e^{2\tau\varphi}.$$

Using again (4.4), we complete the proof of (1.7). □

## 5 Hölder Type Stability in the Cauchy Problem

*Proof of Theorem* 1.3. Since the surface $\Gamma \in C^3$ is noncharacteristic for $\mathbf{A}_R$, we can uniquely solve $\mathbf{A}_R\mathbf{u} = \mathbf{f}$ on $\Gamma$ for $\partial_\nu^2\mathbf{u}$ in terms of $\mathbf{f}$, $\mathbf{g}_0$, $\mathbf{g}_1$, and their tangential derivatives. Moreover,

$$\|\partial_\nu^2\mathbf{u}\|_{(\frac{1}{2})}(\Gamma) \leqslant C(\|f\|_{(\frac{1}{2})}(\Gamma) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)).$$

By extension theorems, for $\mathbf{f} \in H_{(\frac{1}{2})}(\Gamma)$ we can find $\mathbf{u}^* \in H_{(3)}(\Omega)$ such that

$$\mathbf{u}^* = \mathbf{g}_0, \quad \partial_\nu \mathbf{u}^* = \mathbf{g}_1, \quad \partial_\nu^2 \mathbf{u}^* = \partial_\nu^2 \mathbf{u} \ \text{ on } \ \Gamma$$

and

$$\|\mathbf{u}^*\|_{(3)}(\Omega) \leqslant CF. \tag{5.1}$$

Let

$$\mathbf{v} = \mathbf{u} - \mathbf{u}^*. \tag{5.2}$$

The function $\mathbf{v}$ solves the Cauchy problem

$$\mathbf{A}_R \mathbf{v} = \mathbf{f} - \mathbf{A}_R \mathbf{u}^* \ \text{ in } \ \Omega, \quad \mathbf{v} = 0, \quad \partial_\nu \mathbf{v} = 0 \quad \text{on} \quad \Gamma \subset \partial\Omega. \tag{5.3}$$

Moreover, due to our construction of $\mathbf{u}^*$, we have

$$\partial_\nu^2 \mathbf{v} = 0 \ \text{ on } \ \Gamma. \tag{5.4}$$

To apply the Carleman estimates of Theorem 1.2, we need the zero Cauchy data on the whole boundary. To achieve it, we introduce a cut-off function $\chi \in C^\infty(\overline{\Omega})$ such that $\chi = 1$ on $\Omega_{\frac{\delta}{2}}$ and $\chi = 0$ on $\Omega \setminus \Omega_0$. By the Leibniz formula,

$$\mathbf{A}_R(\chi\mathbf{v}) = \chi \mathbf{A}_R \mathbf{v} + \mathbf{A}_1 \mathbf{v},$$

$$\nabla_x \mathbf{A}_R(\chi\mathbf{v}) = \chi \nabla_x \mathbf{A}_R \mathbf{v} + \mathbf{A}_2 \mathbf{v},$$

where $\mathbf{A}_1, \mathbf{A}_2$ are matrix linear partial differential operators with bounded coefficients of orders 1, 2 depending on $\chi$. Moreover, $\mathbf{A}_1 = 0$, $\mathbf{A}_2 = 0$ on $\Omega_{\frac{\delta}{2}}$. Using the Cauchy data (5.3), (5.4), we conclude that $\mathbf{v} \in H^0_{(3)}(\Omega)$. Hence, by the Carleman estimate of Theorem 1.2, we have

$$\int_\Omega (|\nabla_{x,t} \operatorname{div}(\chi\mathbf{v})|^2 + |\nabla_{x,t} \operatorname{curl}(\chi\mathbf{v})|^2 + |\chi\mathbf{v}|^2) e^{2\tau\varphi}$$
$$\leqslant C \int_\Omega (|\mathbf{f}|^2 + |(\mathbf{A}_R\mathbf{u}^*)|^2 + |\nabla_x \mathbf{f}|^2 + |\nabla(\mathbf{A}_R\mathbf{u}^*)|^2 + |\mathbf{A}_1\mathbf{v}|^2 + |\mathbf{A}_2\mathbf{v}|^2) e^{2\tau\varphi}$$

for $C < \gamma, C_0 < \tau$. Shrinking integration domain on the left side to $\Omega_{\frac{3\delta}{4}}$ (where $\chi = 1$) and splitting integration domain of $|\mathbf{A}_2\mathbf{v}|^2$ into $\Omega_{\frac{\delta}{2}}$ and its complement we yield

$$\int_{\Omega_{\frac{3\delta}{4}}} (|\nabla_{x,t} \operatorname{div}(\mathbf{v})|^2 + |\nabla_{x,t} \operatorname{curl}(\mathbf{v})|^2) + |\mathbf{v}|^2) e^{2\tau\varphi}$$
$$\leqslant C \int_\Omega (|\mathbf{f}|^2 + |(\mathbf{A}_R\mathbf{u}^*)|^2 + |\nabla_x \mathbf{f}|^2 + |\nabla_x(\mathbf{A}_R\mathbf{u}^*)|^2) e^{2\tau\varphi}$$

$$+ C \int_{\Omega \setminus \Omega_{\frac{\delta}{2}}} (|\mathbf{A}_1\mathbf{v}|^2 + |\mathbf{A}_2\mathbf{v}|^2) e^{2\tau\varphi} \leqslant CF^2 e^{2\tau\Phi} + C\|\mathbf{v}\|^2_{(2)}(\Omega) e^{2\tau\Phi_2},$$

where we used the definition (1.9) of $F$ and bound (5.1). Let $\Phi = sup\varphi$ over $\Omega$ and $\Phi_2 = sup\varphi$ over $\Omega \setminus \Omega_{\frac{\delta}{2}}$. Letting $\Phi_1 = inf\varphi$ over $\Omega_{\frac{3\delta}{4}}$ and replacing $\varphi$ on the left-hand side of the preceding inequality by $\Phi_1$, we yield

$$(||\mathbf{v}||^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + ||\nabla \operatorname{div} \mathbf{v}||^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + ||\nabla \operatorname{curl} \mathbf{v}||^2_{(1)}(\Omega_{\frac{3\delta}{4}})) e^{2\tau\Phi_1}$$
$$\leqslant CF^2 e^{2\tau\Phi} + C\|\mathbf{v}\|^2_{(2)}(\Omega) e^{2\tau\Phi_2}. \tag{5.5}$$

Note that $\Phi_2 < \Phi_1$.

We remind an interior Schauder type estimate for elliptic equations with zero Dirichlet data on $\Gamma$:

$$\|v\|_{(2)}(\Omega_\delta) \leqslant C\|\Delta v\|_{(0)}(\Omega_{\frac{3\delta}{4}}) + C\|v\|_{(0)}(\Omega_{\frac{3\delta}{4}}).$$

Using, in addition, the equality $\Delta = \nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}$, we obtain

$$||\mathbf{v}||^2_{(2)}(\Omega_\delta) \leqslant C(||\mathbf{v}||^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + ||\nabla \operatorname{div} \mathbf{v}||^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + ||\nabla \operatorname{curl} \mathbf{v}||^2_{(1)}(\Omega_{\frac{3\delta}{4}})).$$

Hence from (5.5) we have

$$||\mathbf{v}||^2_{(2)}(\Omega_\delta) \leqslant CF^2 e^{2\tau(\Phi - \Phi_1)} + C\|\mathbf{v}\|^2_{(2)}(\Omega) e^{2\tau(\Phi_2 - \Phi_1)}. \tag{5.6}$$

If $\|\mathbf{v}\|_{(2)}(\Omega)F^{-1} < C$, then $\|\mathbf{v}\|_{(2)}(\Omega) \leqslant CF$. Otherwise, we let $\tau = (\Phi + \Phi_1 - \Phi_2)^{-1} log(\|\mathbf{v}\|_{(2)}(\Omega)F^{-1})$. Then the bound (5.6) implies

$$||\mathbf{v}||_{(2)}(\Omega_\delta) \leqslant C\|\mathbf{v}\|_{(2)}(\Omega)^{1-\kappa} F^\kappa$$

with $\kappa = \dfrac{\Phi_1 - \Phi_2}{\Phi + \Phi_1 - \Phi_2}$. Combining both cases, we yield

$$||\mathbf{v}||_{(2)}(\Omega_\delta) \leqslant C(F + \|\mathbf{v}\|_{(2)}(\Omega)^{1-\kappa} F^\kappa).$$

Using the equality $\mathbf{u} = \mathbf{v} + \mathbf{u}^*$, the triangle inequality, (5.1), and the elementary inequality $(a+b)^\kappa \leqslant a^\kappa + b^\kappa$, we obtain (1.9) and complete the proof. □

## 6 Lipschitz Stability in the Cauchy Problem

*Proof of Theorem* 1.4. As the proof of Theorem 1.3, we introduce functions $\mathbf{u}^*, \mathbf{v}$ and we use the relations (5.1) and (5.3). We introduce the following energy integrals for the hyperbolic system of elasticity with residual stress:

$$E(t;\mathbf{v}) = \int_G \Big((\partial_t \mathbf{v})^2 + |\nabla \mathbf{v}|^2 + |\mathbf{v}|^2\Big)(,t), \quad E(t) = E(t;\mathbf{v}) + E(t;\nabla\mathbf{v}).$$

Dividing the system (1.8) by $\rho$ and differentiating with respect to the spatial variables, we obtain the extended system with the same principal part

$$\rho^{-1}\mathbf{A}_R\mathbf{v} = \rho^{-1}\mathbf{f}^*,$$

$$\rho^{-1}\mathbf{A}_R\partial_j\mathbf{v} = \partial_j\rho^{-1}\mathbf{f}^* - (\partial_j\rho^{-1}\mathbf{A}_R)\mathbf{v} \text{ in } \Omega = G\times(-T,T), \quad j=1,2,3,$$

where $\mathbf{f}^* = \mathbf{f} - \mathbf{A}_R\mathbf{u}^*$, with the zero boundary value conditions

$$\mathbf{v} = 0, \quad \partial_j\mathbf{v} = 0 \quad \text{on} \quad \Gamma = \partial G\times(-T,T).$$

By standard energy estimates for hyperbolic systems (see, for example, [2]),

$$C^{-1}(E(0) - ||\mathbf{f}^*||_{(1)}(\Omega)) \leqslant E(t) \leqslant C(E(0) + ||\mathbf{f}^*||_{(1)}(\Omega)), \quad t\in(-T,T). \tag{6.1}$$

We choose a smooth cut-off function $0\leqslant\chi(t)\leqslant 1$ such that $\chi_0(t) = 1$ for $-T+2\delta < t < T-2\delta$ and $\chi(t) = 0$ for $|t| > T-\delta$. It is clear that

$$\begin{aligned} \mathbf{A}_R(\chi\mathbf{v}) &= \chi\mathbf{f}^* + 2\rho\partial_t\chi\partial_t\mathbf{v} + \rho\partial_t^2\chi\mathbf{v}, \\ \nabla\mathbf{A}_R(\chi\mathbf{v}) &= \chi\nabla\mathbf{f}^* + 2\rho\partial_t\chi\partial_t\nabla\mathbf{v} + \rho\partial_t^2\chi\nabla\mathbf{v}. \end{aligned} \tag{6.2}$$

As in the proof of Theorem 1.3, $\chi\mathbf{v}\in H^0_{(3)}(\Omega)$. Hence, by Theorem 1.2 (with fixed $\gamma$),

$$\begin{aligned} &\int_\Omega \Big(\sigma(|\nabla_{x,t}(\chi\mathbf{v})|^2 + |\nabla_{x,t}\operatorname{div}(\chi\mathbf{v})|^2 + |\nabla_{x,t}\operatorname{curl}(\chi\mathbf{v})|^2) \\ &+ \sigma^3(|\chi\mathbf{v}|^2 + |\operatorname{div}(\chi\mathbf{v})|^2 + |\operatorname{curl}(\chi\mathbf{v})|^2)\Big)e^{2\tau\varphi} \\ &\leqslant C\int_\Omega (|\mathbf{A}_R(\chi\mathbf{v})|^2 + |\nabla\mathbf{A}_R(\chi\mathbf{v})|^2)e^{2\tau\varphi} \\ &\leqslant C\Big(\int_\Omega (|\mathbf{f}^*|^2 + |\nabla\mathbf{f}^*|^2)e^{2\tau\varphi} + \int_{G\times\{T-2\delta<|t|<T\}} (|\partial_t\mathbf{v}|^2 + |\mathbf{v}|^2 + |\partial_t\nabla\mathbf{v}|^2 + |\nabla\mathbf{v}|^2)e^{2\tau\varphi}\Big) \end{aligned}$$

in view of (5.3).

Shrinking the integration domain $\Omega$ on the left-hand side to $G\times(0,\delta)$ where $\chi = 1$ and choosing $\psi$ by $e^{2\tau(1-\delta)} < e^{2\tau\varphi}$ since $1-\delta < \varphi$ on $G\times(0,\delta)$ and $e^{2\tau\varphi} < e^{2\tau(1-2\delta)}$ since $\varphi < 1-2\delta$ on $G\times(T-\delta,T)$, we have

$$e^{2\tau(1-\delta)}\int_0^{\delta} E(t)dt \leqslant C\Big(\int_{\Omega}(|\mathbf{f}^*|^2+|\nabla\mathbf{f}^*|^2)e^{2\tau\varphi}+Ce^{2\tau(1-2\delta)}$$

$$\times\int_{T-2\delta}^{T}\int_{G}(|\partial_t\mathbf{v}|^2+|\mathbf{v}|^2+|\partial_t\nabla\mathbf{v}|^2+|\nabla\mathbf{v}|^2)\Big).$$

Hence

$$e^{2\tau(1-\delta)}\int_0^{\delta} E(t)dt \leqslant C\Big(\int_{\Omega}(|\mathbf{f}^*|^2+|\nabla\mathbf{f}^*|^2)e^{2\tau\varphi}+Ce^{2\tau(1-2\delta)}\int_{T-2\delta}^{T}E(t)dt\Big).$$

Choosing $\Phi=\sup\limits_{\Omega}\varphi$ and using (6.1), we find

$$e^{2\tau(1-\delta)}\frac{\delta}{C}E(0)-Ce^{2\tau\Phi}||\mathbf{f}^*||^2_{(1)}(\Omega))\leqslant C\delta e^{2\tau(1-2\delta)}E(0)+Ce^{2\tau\Phi}||\mathbf{f}^*||^2_{(1)}(\Omega)).$$

To eliminate the first term on the right-hand side, we choose $\tau$ (depending on $C$) so large that $e^{-2\tau\delta}<\frac{1}{C^2}$ and, using the energy estimates (6.1), we finally get

$$E(t;\mathbf{v})+E(t;\nabla\mathbf{v})\leqslant C||\mathbf{f}^*||_{(1)}(\Omega).$$

As in the proof of Theorem 1.3,

$$\begin{aligned}E(t;\mathbf{u})+E(t;\nabla\mathbf{u})&\leqslant C\Big(||\mathbf{f}^*||_{(1)}(\Omega)+E(t;\mathbf{u}^*)+E(t;\nabla\mathbf{u}^*)\Big)\\&\leqslant C\Big(||\mathbf{f}^*||_{(1)}(\Omega)+||\mathbf{A}_R\mathbf{u}^*||_{(1)}(\Omega)+||\mathbf{u}^*||_{(\frac{5}{2})}(\Gamma)+||\partial_\nu\mathbf{u}^*||_{(\frac{3}{2})}(\Gamma)\Big)\\&\leqslant C\Big(||\mathbf{f}||_{(1)}(\Omega)+||\mathbf{g}_0||_{(\frac{5}{2})}(\Gamma)+||\mathbf{g}_1||_{(\frac{3}{2})}(\Gamma)\Big).\end{aligned}$$

The proof is complete. □

## 7 Conclusion

We believe that the Carleman estimates in Theorem 1.1 can be applied to other important systems of mathematical physics, for example, to transversally isotropic elasticity system and some anisotropic Maxwell systems. We expect that by using an appropriate version of Theorem 1.1 (with norms in Sobolev spaces of negative order), as in [7], one can obtain the Carleman estimate (1.7) without terms with div $\mathbf{u}$ and curl $\mathbf{u}$ on the left-hand side and $\nabla\mathbf{A}_R\mathbf{u}$ on the right-hand side. It is probably challenging to obtain Theorems 1.1 and 1.2 with boundary terms (i.e., without assuming that the functions

**u** are zero at the boundary). However, such a generalization seems feasible and it would be quite useful for applications. In particular, then in Theorems 1.3 and 1.4, one can replace $\|\mathbf{f}\|_{(1)}(\Omega_0) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)$ by $\|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{g}_0\|_{(1)}(\Gamma) + \|\mathbf{g}_1\|_{(0)}(\Gamma)$ and reduce the regularity of $\Gamma$ to $C^2$.

It is realistic to combine the proof of Theorem 1.1 and that of Carleman estimates for general anisotropic operators [9, Sect. 3.2] (including Schrödinger type operators) to obtain such estimates with two large parameters. In particular, one has to adjust the concepts of the principal symbol and the pseudoconvexity condition. Possible applications would imply the uniqueness of continuation, controllability, and inverse problems for anisotropic systems of partial differential equations for plates and shells. Currently, there are no theoretical results in this important area.

Next we will apply Theorems 1.1 and 1.2 to identification of elastic parameters ($\rho, \mu, \lambda$ and $r_{jk}$) with arbitrary residual stress from additional boundary data, as it was done in [12, 13] for small residual stress.

**Acknowledgment.** This work was supported in part by the NSF (grants no. DMS 04-05976 and DMS 07-07734) and Emylou Keith and Betty Dutcher Distinguished Professorship at Wichita State University.

## References

1. Albano, P., Tataru, D.: Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system. Electr. J. Differ. Equ. **2000**, 1-15 (2000)
2. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer-Verlag (1976)
3. Eller, M.: Carleman estimates with a second large parameter. J. Math. Anal. Appl. **249**, 491-514 (2000)
4. Eller, M., Isakov, V.: Carleman estimates with two large parameters and applications. Contemp. Math. **268**, 117-137 (2000)
5. Eller, M., Isakov, V., Nakamura, G., Tataru, D.: Uniqueness and stability in the Cauchy problem for the Maxwell's and elasticity systems. Stud. Appl. Math. **31**, 329-349 (2002)
6. Hörmander, L.: Linear Partial Differential Operators. Springer-Verlag (1963)
7. Imanuvilov, O., Isakov, V., Yamamoto, M.: An inverse problem for the dynamical Lamé system with two sets of boundary data. Commun. Pure Appl. Math. **56**, 1-17 (2003)
8. Isakov, V.: A Nonhyperbolic Cauchy Problem for $\Box_b, \Box_c$ and its Applications to Elasticity Theory. Commun. Pure Appl. Math. **39**, 747-769 (1986)
9. Isakov, V.: On the uniqueness of the continuation for a thermoelasticity system. SIAM J. Math. Anal. **33**, 509-522 (2001)
10. Isakov, V.: Inverse Problems for Partial Differential Equations. Springer-Verlag, New York (2005)
11. Isakov, V., Nakamura, G., Wang, J.-N.: Uniqueness and stability in the Cauchy problem for the elasticity system with residual stress. Contemp. Math. **333**, 99-113 (2003)
12. Isakov, V., Wang, J.-N., Yamamoto, M.: Uniqueness and stability of determining the residual stress by one measurement. Commun. Parital Differ. Equ. **23**, 833-848 (2007)

13. Isakov, V., Wang, J.-N., Yamamoto, M.: An inverse problem for a dynamical Lamé system with residual stress. SIAM J. Math. Anal. **39**, 1328-1343 (2007).
14. Khaidarov, A.: Carleman estimates and inverse problems for second order hyperbolic equations. Math. USSR Sb. **58**, 267-277 (1987)
15. Lasiecka, I., Triggiani, R., Yao, P.F.: Inverse/observability estimates for second order hyperbolic equations with variable coefficients. J. Math. Anal. Appl. **235**, 13-57 (2000)
16. Man, C.-S.: Hartig's law and linear elasticity with initial stress. Inverse Problems **14**, 313-320 (1998)
17. Rachelle, L.: Uniqueness in Inverse Problems for Elastic Media with Residual Stress. Commun. Parital Differ. Equ. **28**, 1787-1806 (2003)
18. Romanov, V.: Carleman estimates for second-order hyperbolic equations. Sib. Math. J. **47**, 135-151 (2006)

# Sharp Spectral Asymptotics for Dirac Energy

Victor Ivrii

*To the Memory of my Teacher Sergey Sobolev*

**Abstract** I derive sharp semiclassical asymptotics of

$$\int |e_h(x,y,0)|^2 \omega(x,y)\, dx\, dy,$$

where $e_h(x,y,\tau)$ is the Schwartz kernel of the spectral projector and $\omega(x,y)$ is singular as $x=y$. I also consider asymptotics of more general expressions.

## 1 Introduction

In the series of papers [7, 3, 5, 4] devoted to the sharp asymptotics of the ground state energy of heavy atoms and molecules, it was needed to calculate *Dirac correction term*[1] which in that approximation was equal to

$$I \stackrel{\text{def}}{=} \iint |e(x,y,\tau)|^2 |x-y|^{-1}\, dx\, dy, \tag{1.1}$$

where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of the (magnetic) Schrödinger operator

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[1] Representing Coulomb interaction of electrons with themselves which should not to be counted in the energy calculation and should be subtracted from the Thomas–Fermi expression.

V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series.*

doi: 10.1007/978-0-387-85652-0, 

$$A = \frac{1}{2}\Bigl(\sum_{j,k} P_j g^{jk}(x) P_k - V\Bigr), \qquad P_j = hD_j - \mu V_j, \tag{1.2}$$

$\tau \approx 0$ and $h \to +0$ (while either $\mu \to +\infty$ or remains constant). Actually, the corresponding part of these papers was originally more complicated, but it was reduced to the problem above.

Then $I \asymp h^{-d-1}$, where $d$ is the dimension ($d = 3$ in the above papers), and it was needed to prove that $I = \mathcal{I} + O(h^{-d-1+\delta})$ with $\mathcal{I}$ defined by the same formula, but with $e(x, y, \tau)$ replaced by

$$e^{\mathrm{W}}_y(x, y, \tau) \stackrel{\mathrm{def}}{=} (2\pi h)^{-d} \int\limits_{g(y,\xi)\leqslant V(y)+2\tau} e^{ih^{-1}\langle x-y,\xi\rangle}\, d\xi \tag{1.3}$$

and with a small exponent $\delta > 0$; for the magnetic Schrödinger operator it was needed to prove as $\mu \leqslant h^{-\delta}$ only. The expression (1.3) is a *Weyl expression* for $e(x, y, \tau)$ for operator with coefficients frozen at point $y$.

However, I believe that the asymptotics of the expression (1.1) or more general one is interesting by itself and that there are sharp asymptotics. Still my attempts to derive it were not very successful and, in [2], I made some claims which I could not sustain at that time. So, in this paper, I just want to bring some degree of the order to this matter.

I am going to consider a matrix $h$-differential operator $A(x, hD)$ and find asymptotics of

$$I \stackrel{\mathrm{def}}{=} \iint \omega(x, y) e(x, y, \tau)\psi_2(x) e(y, x, \tau)\psi_1(y)\, dx\, dy \tag{1.4}$$

with a matrix-valued function $\omega(x, y)$ such that

$$\begin{gathered}\omega(x, y) \stackrel{\mathrm{def}}{=} \Omega(x, y; x - y), \text{ where the function } \Omega \text{ is smooth} \\ \text{in } B(0, 1) \times B(0, 1) \times B(\mathbb{R}^d \setminus 0) \text{ and homogeneous of degree} \\ -\kappa\ (0 < \kappa < d) \text{ with respect to its third argument}^2\end{gathered} \tag{1.5}$$

and with smooth cut-off functions $\psi_1, \psi_2$.

The main part of asymptotics should have a magnitude of $h^{-d-\kappa}$, and I would like to get a remainder estimate $O(h^{1-d-\kappa})$.

One can also consider a more general expression

$$\begin{aligned}I_m \stackrel{\mathrm{def}}{=} \iint & \omega(x^1, \ldots, x^m) e(x^1, x^2, \tau)\psi_2(x^1) \\ & \times e(x^2, x^3, \tau) \cdots e(x^m, x^1, \tau)\psi_{m+1}(x^0) dx^1 \cdots dx^m\end{aligned} \tag{1.6}$$

[2] In other words, it is the Michlin–Calderon–Zygmund kernel.

with $x^{m+1}=x^1$, $\psi_{m+1}\stackrel{\mathrm{def}}{=}\psi_1$ etc., and

$\omega(x^1,\dots,x^m)\stackrel{\mathrm{def}}{=}\Omega(x^1,\dots,x^m;\{x^j-x^{j+1}\}_{1\leqslant j\leqslant m})$, where the function $\Omega$ is smooth in $B(0,1)^m\times B(\mathbb{R}^d\setminus 0)^{m-1}$ and homogeneous of degree $-(m-1)\kappa$ with respect to $\{x^j-x^k\}_{1\leqslant j<k\leqslant m}$. Moreover,

$$|D_{\mathsf{z}}^{\beta}D_{\mathsf{x}}^{\nu}\Omega|\leqslant C_{\beta,\nu}|z^1|^{-\kappa-|\beta^1|}\cdots|z^m|^{-\kappa-|\beta^m|} \tag{1.7}$$
$$\text{as}\sum_k|z^k|^2=1,\ \sum_k z^k=0,$$

where $\mathbf{x}=(x^1,\dots,x^m)$, $\mathbf{z}=(z^1,\dots,z^{m-1})$, etc.

However, I will leave it for another paper since not of all my arguments I was able to implement in this case.

The main part of asymptotics should have a magnitude of $h^{-d-(m-1)\kappa}$ (see Theorem 2.6), and I would like to get a remainder estimate $O(h^{1-d-(m-1)\kappa})$.

I am also leaving for another paper a similar, but much more delicate and difficult analysis for the 2-dimensional magnetic Schrödinger operator(1.2) with trajectories having many loops.

**Remark 1.1.** (i) To avoid the necessity to cut-off with respect to $hD$, one needs to assume that its symbol satisfies

$$|a(x,\xi)|^{-1}\leqslant C|\xi|^{-m}\qquad \text{as } |\xi|\geqslant C_0 \tag{1.8}$$

as $a\in\Psi^m$ (one can weaken this condition, but I leave it to the reader).

(ii) One needs to assume that $a$ is semibounded from below which under (1.8) is equivalent to

$$\langle a(x,\xi)v,v\rangle\geqslant c^{-1}|v|^2\qquad \text{as } |\xi|\geqslant C_0; \tag{1.9}$$

otherwise, instead of $E(\tau)$, one should consider $E(\tau_1,\tau_2)\stackrel{\mathrm{def}}{=}E(\tau_2)-E(\tau_1)$; I leave it to the reader as well.

The problem studied here is not exactly microlocal due to singular kernel $\omega$. However microlocal methods (see f.e. [6]) play a crucial role. Microlocal methods telling us where and in what direction distribution belongs to Sobolev space $H^s$. The other application of Sobolev spaces is more explicit since we need to apply imbedding theorems just to estimate properly contribution of zone near diagonal $x=y$.

This paper consist of three sections. In Sect. 2, I derive asymptotics with the sharp remainder estimate, but with the implicit Tauberian approximation for $e(x,y,0)$. In Sect. 3, I replace it by the expression (1.3) without deteriorating the remainder estimate for scalar operators under mild nondegeneracy condition (Theorem 3.19) and for certain matrix operators (Theorem 3.20(i)) and with some not sharp remainder estimates for other matrix operators (Theorem 3.20(ii)). I just mention that for larger $\kappa$ we need less restrictive conditions to operator.

# 2 Estimates

## *2.1 Special case*

Let us assume first that $\omega = 1$ but relax conditions to $\psi_1, \dots, \psi_m$, assuming only that $\psi_1, \dots, \psi_m \in L^\infty$. This is definitely not the case I am interested in, but one needs to make few clarifications first. Then

$$I_m \stackrel{\mathrm{def}}{=} \mathrm{Tr} E(\tau)\psi_2 E(\tau)\psi_3 E_(\tau) \cdots E(\tau)\psi_{m+1} \tag{2.1}$$

containing $m$ factors $E(\tau)$.

Under the condition (1.9), it is known (see, for example, [6]) that if the $L^\infty$ norms and diameters of supports of $\psi$ and $\psi_1$ are bounded, then

$$\|\psi E(\tau)\psi_1\|_1 \leqslant Ch^{-d} \qquad \text{as } |\tau| \leqslant c, \tag{2.2}$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the operator and trace norms respectively. Then, since the operator norm of $E(\tau)$ does not exceed 1, I conclude that $|I_m| \leqslant ch^{-d}$. So,

$$\text{If } \psi_j \in L^\infty \text{ and } I_m \text{ is given by (2.1), then } |I_m| \leqslant Ch^{-d}. \tag{2.3}$$

Further, let us assume that

$$a(x,\xi) \text{ is microhyperbolic on energy level } 0. \tag{2.4}$$

Then as (2.4) is fulfilled on supports of $\psi$, $\psi_1$, it is known (see, for example, [6]) that

$$\|\psi\big(E(\tau) - E(\tau')\big)\psi_1\| \leqslant C(|\tau - \tau'| + hT^{-1})h^{-d} \quad \text{as } |\tau| \leqslant \varepsilon_1,\ |\tau'| \leqslant c. \tag{2.5}$$

Here and for a while, $T \asymp 1$, but I want to keep a track of it.

Since this property holds under wider assumptions than microhyperbolicity, I will assume so far only that (2.5) holds.

Then

$$\Bigl|\mathrm{Tr}'\Bigl(\bigl(E(\tau)-E(\tau')\bigr)\psi_2E(\tau_2)\psi_3E(\tau_3)\cdots E(\tau_m)\psi_{m+1}\Bigr)\Bigr| \tag{2.6}$$

does not exceed the right-hand expression of (2.5) either, as $|\tau|\leqslant\varepsilon_1$ and therefore, due to the standard Tauberian arguments (second part; see, for example, [6]) the following inequality holds:

$$\begin{aligned}&\Bigl|\mathrm{Tr}'\Bigl(\Bigl(E(0)-h^{-1}\int\limits_{-\infty}^{0}F_{t\to h^{-1}\tau}\bigl(\overline{\chi}_T(t)U(t)\bigr)\,d\tau\Bigr)\psi_2E(\tau_2)\\ &\times\psi_3E(\tau_3)\cdots E(\tau_m)\psi_{m+1}\Bigr)\Bigr|\leqslant CT^{-1}h^{1-d},\end{aligned} \tag{2.7}$$

where I use my standard notation $\overline{\chi}$ and $\chi$ in the future and $\overline{\chi}(t)=\overline{\chi}(t/T)$ etc. (see, for example, [1]). Here and below, $\mathrm{Tr}'$ is the "scalar trace" of an operator and does not include taking the matrix trace tr.

Here and below, $U(t)=e^{ih^{-1}tA}$ is the propagator of $A$ and $u(x,y,t)$ is its Schwartz' kernel.

So, with $O(T^{-1}h^{1-d})$ error one could replace one copy of $E(0)$ in $I_m$ by its standard implicit Tauberian approximation

$$h^{-1}\int\limits_{-\infty}^{0}F_{t\to h^{-1}\tau}\bigl(\overline{\chi}_T(t)U(t)\bigr)\,d\tau \tag{2.8}$$

and in by the virtues of the same arguments, I can do it with another copy of $E(0)$. Therefore,

**Proposition 2.1.** *Under the condition* (2.5)*, with an error* $O(T^{-1}h^{1-d})$ $I_m$ *is equal to*

$$h^{-m}\mathrm{Tr}'\int\limits_{\boldsymbol{\tau}\in\mathbb{R}^{-,m}}F_{\mathbf{t}\to h^{-1}\boldsymbol{\tau}}\Bigl(\overline{\chi}_T(t_1)U(t_1)\psi_2\overline{\chi}_T(t_2)U(t_2)\psi_3\cdots U(t_m)\psi_{m+1})\Bigr)\,d\boldsymbol{\tau} \tag{2.9}$$

*with* $\mathbf{t}=(t_1,\dots,t_m)$*,* $\boldsymbol{\tau}=(\tau_1,\dots,\tau_m)$*.*

Note that here one can take any $T\in[Ch^{1-\delta},c]$ (but then the error depends on $T$). Further, note that as $\mathrm{dist}(\mathrm{supp}\psi_j,\mathrm{supp}\psi_{j+1})\geqslant(c_0+\varepsilon)T$, where here and below $c_0$ is the upper bound of the propagation speed on energy level 0 and $x^{m+1}\overset{\mathrm{def}}{=}x^1$, the expression (2.9) as $m=2$ or a similar expression as $m\geqslant 3$ become negligible, and I arrive at

**Corollary 2.2.** *If, in the frames of Proposition* 2.1, $\mathrm{dist}(\mathrm{supp}\psi_j,\mathrm{supp}\psi_{j+1})\geqslant(c_0+\varepsilon)T$ *for some* $j=1,\dots,m$*, then* $|I_m|$ *does not exceed* $CT^{-1}h^{1-d}$.

## 2.2 Smooth case

The next step is to assume that $\omega$ is a smooth function. Without any loss of generality, one can assume that $\omega$ is also compactly supported (since $\psi, \psi_1$ are). Then from

$$\begin{aligned}\omega(x^1,\dots,x^m) &= \int \omega(y^1,\dots,y^m)\delta(y^1-x^1,\dots,y^m-x^m)\,dy\\ &= \int \omega'(y^1,\dots,y^m)\theta(y^1-x^1)\cdots\theta(y^m-x^m)\,dy^1\cdots dy^m \end{aligned}\tag{2.10}$$

one arrives at

$$I_m = \int \omega'(y^1,\dots,y^m)J_2(y^1,\dots,y^m)\,dy^1\cdots dy^m \tag{2.11}$$

with $J_2(y^1,\dots,y^m)$ defined by $\omega = 1$ and $\psi_j(x)$ redefined as $\psi_j(x)\theta(y^j-x)$, where here and below $\theta(x)=\theta(x_1)\cdots\theta(x_d)$. Then I immediately arrive at

**Proposition 2.3.** *Let $\omega$ and $\psi_1,\dots,\psi_m$ be smooth functions, and let the condition* (1.9) *be fulfilled. Then $|I_m|\leqslant Ch^{-d}$.*

**Remark 2.4.** As $m=2$ and $\omega,\psi_1,\psi_2\in L^\infty$ $|I_2|\leqslant Ch^{-d}$ obviously (it follows from the estimate $\|\psi E\psi\|_2 \leqslant Ch^{-d/2}$, where $\|\cdot\|_2$ is the Hilbert–Schmidt norm). Can one prove the similar result for $m\geqslant 3$?

**Proposition 2.5.** *Let $\omega$ and $\psi_1,\dots,\psi_m$ be smooth functions, and let the conditions* (1.9) *and* (2.5) *be fulfilled. Then*

(i) *with an error $O(T^{-1}h^{1-d})$ $I_m$ is equal to*

$$\begin{aligned}\mathcal{I}_m = h^{-m}\int\int_{\boldsymbol{\tau}\in\mathbb{R}^{-,m}} &\omega(x^1,\dots,x^m)F_{\mathbf{t}\to h^{-1}\boldsymbol{\tau}}\Big(\overline{\chi}_T(t_1)u(x^1,x^2,t_1)\psi_2(x^2)\\ &\times\overline{\chi}_T(t_2)u(x^2,x^3,t_2)\psi_3(x^3)\cdots U(t_m)\psi_{m+1}(x^{m+1})\Big)\,d\tau\,dx^1\cdots dx^m\end{aligned}\tag{2.12}$$

*with $x^{m+1}\stackrel{\mathrm{def}}{=}x^1$.*

(ii) *Further, if* $\operatorname{dist}(\operatorname{supp}\psi_j,\operatorname{supp}\psi_{j+1})\geqslant(c_0+\varepsilon)T$ *for some $j=1,\dots,m$, then $|I_m|$ does not exceed $CT^{-1}h^{1-d}$, where so far $T\asymp 1$.*

## 2.3 Singular homogeneous case

**Theorem 2.6.** *Let the conditions* (1.9) *and* (1.7) *be fulfilled. Then $|I_m|\leqslant Ch^{-d-(m-1)\kappa}$.*

*Proof.* Let us replace $\Omega(\mathbf{x},\mathbf{z})$ by $\Omega(\mathbf{x},\mathbf{z})\beta(z^1/\gamma_1)\cdots\beta(z^m/\gamma_m)$, where $\gamma_j\geqslant h$ and $\beta,\overline{\beta}$ are functions (on $\mathbb{R}^d$) similar to $\chi,\overline{\chi}$ respectively. Then, similarly to the analysis of the smooth case, one can estimate the contribution of such a partition element to $I_m$ by

$$Ch^{-d}\bigl(\gamma_1\cdots\gamma_m\bigr)^{-1}(\gamma_1+\cdots+\gamma_m)^{1-\kappa} \tag{2.13}$$

and the summation with respect to $\gamma_j\geqslant\overline{\gamma}=h$ results in the value of this expression as $\gamma_j=\overline{\gamma}$ and the total estimate becomes what is claimed.

However, one needs to consider the other partition elements when some of $\beta(z^j/\gamma_j)$ are replaced by $\overline{\beta}(z^j/\overline{\gamma})$. So we get "sandwiches" consisting of the factors

$$e(x^k,x^{k+1},\tau)\beta(z^{k+1}/\gamma_{k+1})\cdots\beta(z^j/\gamma_j)e(x^j,x^{j+1},\tau)$$

with $j\geqslant k$ and in between them the factors $\overline{\beta}(z^k/\overline{\gamma})$.

Let $J$ be the set of indices appearing in $\overline{\beta}(z^k/\overline{\gamma})$ (for a given type of a "sandwich"). One can see easily that the contribution of each "sandwich" to $I_m$ does not exceed

$$Ch^{-dr}\prod_{j\notin J}\gamma_j^{-\kappa}\times\Bigl(\int_{\{|z|\leqslant\overline{\gamma}\}}|z|^{-\kappa}\,dz\Bigr)^{r-1}\asymp Ch^{-dr}\prod_{j\notin J}\gamma_j^{-\kappa}\times\overline{\gamma}^{(d-\kappa)(r-1)},$$

where $r$ is the number of factors of each type. Then, after the summation with respect to $\gamma_j\geqslant\overline{\gamma}$, one gets the same expression with $\gamma_j=\overline{\gamma}$, i.e., $Ch^{-dr}\overline{\gamma}^{\kappa(m-r)+(d-\kappa)(r-1)}=Ch^{-dr}\overline{\gamma}^{-\kappa(m-1)+d(r-1)}$ which is exactly what we want as $\overline{\gamma}\asymp h$. □

It immediately follows from the proof of a stronger condition

**Proposition 2.7.** *Let the conditions* (1.9) *and* (1.7) *be fulfilled. Then replacing* $\Omega(\mathbf{x},\mathbf{z})$ *by* $\Omega(\mathbf{x},\mathbf{z})\overline{\beta}(z^1/\gamma)\cdots\overline{\beta}(z^m/\gamma)$ *results in the error not exceeding*

$$Ch^{-d-(m-2)\kappa}\gamma^{-\kappa}. \tag{2.14}$$

Now, let us assume, instead of the condition (2.4) or (2.5), that

$$\text{$a(x,\xi)$ is microhyperbolic on the energy level 0 and microhyperbolicity directions are (at each point) $\ell_\xi\cdot\partial_\xi$ [3] with $\ell_\xi=\ell_\xi(x,\xi)$.} \tag{2.15}$$

**Proposition 2.8.** *Let the conditions* (1.9), (1.7), *and* (2.15) *be fulfilled. Then replacing* $\Omega(\mathbf{x},\mathbf{z})$ *by* $\Omega(\mathbf{x},\mathbf{z})\overline{\beta}(z^1/\gamma)\cdots\overline{\beta}(z^m/\gamma)$ *results in the error not exceeding*

$$Ch^{1-d-(m-2)\kappa}\gamma^{-1-\kappa}. \tag{2.16}$$

[3] So, $\ell_x=0$.

*This is equivalent to taking* $T \asymp \gamma$ *in* (2.8) *and plugging the Schwartz kernel of it instead of* $e(x,y,0)$ *in the definition of* $I_m$.

*Proof.* Proof follows from the combined arguments of the proofs of Theorem 2.6 and Proposition 2.1; in this case, one needs to consider only "sandwiches" containing at least one factor $\beta(x^j/\gamma_j)$ with $\gamma_j \geqslant \gamma$ which accounts for a factor $h/\gamma_j$ and the summation with respect to partition results in an extra factor $h/\gamma$. □

So, one needs to study the expression (2.12) with some $T = T^*$; I remind that the remainder estimate contains the factor $T^{*\,-1}$. One can decompose $\overline{\chi}_{T^*}(t)$ into the sum of $\overline{\chi}_{\overline{T}}(t)$ and $\chi_T(t)$ with $T$ running between $\overline{T}$ and $T^*$ and also one can take $\overline{T} = Ch$. Then the expression (2.12) becomes the sum of similar expressions with $\overline{\chi}_T(t)$ (with $T = T^*$) replaced by $\phi_{jT_j}(t)$ where *either* $\phi_j = \chi$ and $\overline{T} \leqslant T_j \leqslant T^*$ *or* $\phi_j = \overline{\chi}$ and $T_j = \overline{T}$.

In this expression, as $\phi_j = \chi$ one can replace $\int\limits_{-\infty}^{0} (\dots)\, d\tau$ by $(\dots)|_{\tau=0}$ simultaneously replacing $h^{-1}\chi_T(t)$ by $it^{-1}\chi_T(t) = T^{-1}\phi_T(t)$ with $\phi(t) = it^{-1}\chi(t)$; so we get a modified expression (2.12) with $r$ factors $\overline{\chi}_{\overline{T}}(t_j)$ and $\tau_j$ snapped to 0 for $j \in J$, $r = \#J$ and the integration over $\mathbb{R}^{-\,(m-r)}$ and $(m-r)$ factors $\phi_T(t_k)$, $k \notin J$; furthermore, the factor $h^{-m}$ is replaced by $h^{-r}\prod_{k\notin J} T_k^{-1}$.

**Proposition 2.9.** *Let the conditions* (1.9) *and* (2.15) *be fulfilled, and let* $\omega$ *be a smooth function,*

$$\omega = O\bigl((|x^1 - x^2| + \cdots + |x^m - x^1|)^K\bigr). \tag{2.17}$$

*Then* $I_m = O(h^{1-d})$ *as* $K > 1$ *and* $I_m = O(h^{1-d}|\log h|)$ *as* $K = 1$.

*Proof.* Proof follows from the combined arguments of the proofs of Theorem 2.6 and Proposition 2.1 like in Proposition 2.8. Here, however, the main contribution (as $K \geqslant 1$) is delivered by the zone $\{|x^1 - x^2| + \cdots + |x^m - x^1| \asymp 1\}$. □

One can consider certain generalizations, but I will do it later.

## 3 Calculations

Now, our purpose is to go from the implicit Tauberian expression (2.12) to a more explicit one.

## *3.1 Constant coefficients case*

Let us first consider the case $A(x,\xi)=A(\xi)$. In this case,

$$e(x,y,\tau)=(2\pi h)^{-d}\int e^{ih^{-1}\langle x-y,\xi\rangle}E(\xi)\,d\xi, \tag{3.1}$$

where $E(\xi,\tau)$ is the matrix projector corresponding to $A(\xi)$. Then

$$\begin{aligned} I_m=(2\pi h)^{-dm}\int\int \omega(x^1,\dots,x^m)E(\xi^1,0)\cdots E(\xi^m,0)\\ \times e^{ih^{-1}\bigl(\langle x^1-x^2,\xi^1\rangle+\langle x^2-x^3,\xi^2\rangle+\dots+\langle x^m-x^1,\xi^m\rangle\bigr)}\,dx^1\cdots dx^m\,d\xi^1\cdots d\xi^m. \end{aligned} \tag{3.2}$$

From now and until the end of the paper I am assuming that $m=2$. (3.3)

Without any loss of generality, one can assume that either $\omega(x,y)$ is of the form

$$\omega(x,y)=\Omega\Bigl(\frac{1}{2}(x+y),x-y\Bigr). \tag{3.4}$$

or it is of the same singular type as before, but multiplied by $(x_k-y_k)$. However, in the latter case (under microhyperbolicity condition), one can apply a Tauberian approximation for $e(x,y,\tau)$ equal 0 with the remainder estimate $O(h^{1-d}|x-y|^{-1})$ (in the same trace class as before) which leads to $I\approx 0$ with the sought remainder estimate $O(h^{1-d-\kappa})$.

In the former case (3.4), we get

$$I\overset{\mathrm{def}}{=}I_2=\int \mathcal{J}(x)\,dx, \tag{3.5}$$

where

$$\begin{aligned} \mathcal{J}(x)&=2(2\pi h)^{-2d}\iiint \Omega(x,z)E(\xi,0)E(\eta,0)e^{ih^{-1}\langle z,\xi-\eta\rangle}\,dzd\xi d\eta\\ &=G(x)h^{-d-\kappa}, \end{aligned} \tag{3.6}$$

with

$$G(x)=\iint \widehat{\Omega}(x,\xi-\eta)E(\xi,0)E(\eta,0)\,d\xi d\eta, \tag{3.7}$$

and

$$\widehat{\Omega}(x,\zeta)=2(2\pi)^{-2d}\int \Omega(x,z)e^{i\langle z,\zeta\rangle}\,dz. \tag{3.8}$$

One always can take $\Omega$ having compact support with respect to $x$ (since we had originally cutoffs $\psi_1(x^1),\dots,\psi_m(x^m)$).

**Remark 3.1.** (i) One can easily generalize (3.5)–(3.8) to $m>2$.
(ii) The integral (3.8) converges as $|z|\leqslant 1$ since $\kappa<d$. On the other hand, it defines a distribution with respect to $\zeta$ which is positively homogeneous of degree $\kappa-d$ and also is smooth as $\zeta\neq 0$; thus, $\widehat{\Omega}\in L^1_{\mathrm{loc}}$ and (3.7) is well defined. However, generalization to $m>2$ is not that easy.

## *3.2 General microhyperbolic case*

Note first that, due to the microhyperbolicity condition (2.15), one should take $T\asymp\gamma$ as $m=2$ [4]. Otherwise, as $T\in[Ch^{1-\delta},T^*]$, $T^*$ is a small constant, the contribution of $[T/2,T]\cup[-T,-T/2]$ would be negligible.

To calculate $u$, let us apply the successive approximation method on the time interval $[-T,T]$ with $h^{1-\delta}\leqslant T$. Then, plugging the successive approximation into any copy on that interval, we arrive at an error in $u$ in the trace norm equal to $O(h^{-d}(T^2/h)^n)$, where $n$ is the number of the first dropped term (starting from 0). This leads to the error in $I$ $O\bigl(h^{-d-\kappa}(T^2/h)^n\gamma^{-\kappa}\bigr)$ as $T\geqslant\gamma$. Since, under the microhyperbolicity assumption (2.15), we need to consider only $T\asymp\gamma$, the error is $O(h^{-d}(T^2/h)^nT^{-\kappa})$. However, if we just take $u=0$, then we get an error $O(h^{1-d}T^{-1-\kappa})$.

Finding $T$ from the equation

$$h^{-d}(T^2/h)^n=h^{1-d}T^{-1},$$

we get

$$T=h^{(n+1)/(2n+1)} \tag{3.9}$$

(which is greater than $h^{1-\delta}$ with $\delta>0$) and this leads to an error

$$O\bigl(h^{1-d-(n+1)(\kappa+1)/(2n+1)}\bigr). \tag{3.10}$$

**Proposition 3.2.** *Let the conditions* (1.9), (1.7), *and* (2.15) *be fulfilled. Then*

(i) *Using successive approximation as* $|t|\leqslant T$ *given by* (3.2) *and taking* $u=0$ *otherwise, we get* $I$ *with an error given by* (3.10).

(ii) *In particular, this is the sharp remainder estimate* $O(h^{1-d-\kappa})$ *as*

$$\kappa\geqslant(n+1)/n; \tag{3.11}$$

*in particular, as* $\kappa\geqslant 2$, *one can skip all perturbation terms and get the same answer* (3.4)–(3.7).

[4] And $T_j\asymp|x^j-x^{j+1}|$ in the general case.

On the other hand, if we cannot skip some term, then this is given by the same formulas (3.4)–(3.7) as before, but with the factor $h^{-d-\kappa+s}$ instead of $h^{-d-\kappa}$ and with $\Omega$ replaced by $\Omega_s$ positively homogeneous of degree $-\kappa+s$ (provided that these formulas have sense!). Then, as long as $s<\kappa$, one can see that these terms are less than the remainder estimate and we arrive at

**Proposition 3.3.** *Let the conditions* (1.9), (1.7), *and* (2.15) *be fulfilled. Then*

(i) *As* $\kappa>1$, *formulas* (3.4)–(3.7) *provide an answer with the remainder estimate* $O(h^{1-d-\kappa})$.

(ii) *As* $\kappa\leqslant 1$, *formulas* (3.4)–(3.7) *provide an answer with the remainder estimate* $O(h^{\frac{1}{2}(1+\kappa)-d-\kappa-\delta})$ *with an arbitrarily small exponent* $\delta>0$.

## *3.3 Scalar case*

Let us completely analyze the case of a scalar operator $A$.

**3.3.1** Assume first that $\omega=1$ and $\psi_1,\psi_2$ are smooth functions. Then one can rewrite (2.9) with $m=2$

$$h^{-2}\mathrm{Tr}\int\limits_{(\tau_1,\tau_2)\in\mathbb{R}^{-,2}} F_{t_1\to h^{-1}\tau_1,t_2\to h^{-1}\tau_2}\Bigl(\overline{\chi}_T(t_1)\overline{\chi}_T(t_2)\psi_1U(t_1)\psi_2U(t_2)\Bigr)\,d\tau \tag{3.12}$$

with $T=T^*$ which is the largest value for which the remainder estimate $O(T^{-1}h^{1-d})$ for the standard asymptotics was derived; here, $T^*\asymp 1$.

If we replace some copies of $\overline{\chi}_T(t_k)$ by $\chi_{T_k}(t_k)$ with $Ch\leqslant T_k\leqslant T^*$, then one can replace also the operator $h^{-1}\int\limits_{-\infty}^{0}(\dots)\,d\tau_k$ by $T^{-1}(\dots)|_{\tau_k=0}$ and $\chi$ by $it^{-1}\chi$.

If we do it with both $k=1,2$, then we get a term $O(h^{-d})$ (the better estimate is actually possible) and the summation with respect to all partitions with respect to $T_1,T_2$ results in $O(h^{-d}|\log h|^2)$ which differs from the proper estimate by $|\log h|^2$ factor. If we replace some copies of $\overline{\chi}_T(t_k)$ by $\overline{\chi}_h(t_k)$, then we do not make a transformation with respect to these factors, but we gain a factor $h$ due to the size of the support. So, after the summation with respect to partition, we arrive at estimate $O(h^{-d}|\log h|^{2-r})$ for $I$, where $r$ is the number of $\overline{\chi}_h(t_k)$ factors.

On the other hand, the expression (3.12) is equal to

$$h^{-2}\mathrm{Tr}\int\limits_{(\tau_1,\tau_2)\in\mathbb{R}^{-,2}} F_{t_1\to h^{-1}\tau_1,t_2\to h^{-1}\tau_2}\Bigl(\overline{\chi}_T(t_1)\overline{\chi}_T(t_2)\psi_1\psi_{2,t_1}U(t_1+t_2))\Bigr)\,d\tau \tag{3.13}$$

with $\psi_t = U(t)\psi U(-t)$.

Applying a standard approach, we arrive at

$$\mathcal{I} \sim \sum_{n\geqslant 0} \varkappa_n h^{-d+n}, \tag{3.14}$$

where $\mathcal{I} = \mathcal{I}_2$ is defined by (2.12).

Let us replace in (3.13) $\psi_{2,t_1}$ by $\psi_2$. Plugging $t_{1,2} = \frac{1}{2}t \pm z$, $\tau_{1,2} = \tau \pm \tau'$, we arrive at

$$h^{-1}\mathrm{Tr}\int\limits_{\infty}^{0} \Bigl(\int\limits_{\mathbb{R}} \rho_T(t,\tau)\psi_1\psi_2 U(t) e^{-ih^{-1}t\tau}\, dt\Bigr)\, d\tau, \tag{3.15}$$

where $\rho_T(t,\tau) = \rho(t/T,\tau)$, $\tau < 0$,

$$\rho(t,\tau) = -\pi^{-1}h^{-1}\int\limits_{\mathbb{R}} \overline{\chi}_T(\frac{1}{2}t+z)\overline{\chi}_T(\frac{1}{2}t-z)z^{-1}\sin(h^{-1}Tz\tau)\, dz \tag{3.16}$$

is $C_0^\infty([-2,2])$ and one can prove easily that

$$|\partial_t^n\bigl(\rho(t,\tau) \mp \overline{\chi}^2(t/2)\bigr)| \leqslant C_{nm}(1+|\tau|Th^{-1})^{-m} \qquad \forall m,n\ \forall \tau \lesseqgtr 0. \tag{3.17}$$

Then, due to (3.17), only the zone $\{|\tau| \leqslant h^{1-\delta}\}$ gives a nonnegligible contribution to this error and due to the microhyperbolicity condition there $|\mathrm{Tr}\psi_1\psi_2 U(t)| \leqslant Ch^{-d}(1+|t|h^{-1})^{-m}$ which, together with (3.17), implies

Under the microhyperbolicity condition (2.4), the expression (3.15) is equal modulo $O(h^{1-d})$ to the same expression with $\rho$ replaced by $\overline{\chi}^2(t/2)$. (3.18)

On the other hand, if we replace $\psi_{2,t_1}$ by $\psi_{2,t_1} - \psi_2 = t_1\psi'_{2,t_1}$, then we can apply the same transformation as before just getting rid of one factor $h^{-1}$ and the integration with respect to $\tau_1$, which simply snaps to 0, resulting in expression, similar to (3.15), but with $\rho\psi_2$ replaced by

$$\rho'(t,\tau,x) = (2\pi)^{-1} i \int\limits_{\mathbb{R}} \overline{\chi}_T(z)\overline{\chi}_T(t-z)e^{ih^{-1}\tau z}\psi'_{2,z}\, dz \tag{3.19}$$

which satisfies an inequality similar to (3.17)

$$|\partial_t^n\rho(t,\tau)| \leqslant C_{nm}(1+|\tau|Th^{-1})^{-m} \qquad \forall m,n\ \forall \tau \lesseqgtr 0. \tag{3.20}$$

and therefore,

Under the microhyperbolicity condition (2.4), this new (3.15)-type expression is $O(h^{1-d})$. (3.21)

So, we are left with the expression (3.15) with $\rho(t) = \overline{\chi}^2(t/2)$, but due to the standard theory, we get modulo $O(h^{1-d})$ the expression

$$\mathrm{Tr}\psi_1\psi_2 E(0) \equiv (2\pi h)^{-d} \iint_{\{a(x,\xi)<0\}} \psi_1\psi_2\,dx\,d\xi. \tag{3.22}$$

So, $\mathcal{I}$ is given by (3.22) modulo $O(h^{1-d})$ and therefore,

$$\varkappa_0 = (2\pi)^{-d} \int \iint_{\{a(x,\xi)<0\}} \psi_1\psi_2\,dx\,d\xi \qquad \text{in (3.14)}. \tag{3.23}$$

**3.3.2** Then, in the general smooth case, we get

**Proposition 3.4.** *Let $\omega$ and $\psi_1,\dots,\psi_m$ be smooth functions, and let* (1.9) *and the microhyperbolicity condition* (2.4) *be fulfilled. Then with an error $O(T^{-1}h^{1-d})$, where $T \asymp 1$ here, the decomposition* (3.14) *holds with*

$$\varkappa_0 = (2\pi)^{-d} \iint_{\{a(x,\xi)<0\}} \omega(x,x)\psi_1(x)\psi_2(x)\,dx\,d\xi. \tag{3.24}$$

*Proof.* The proof follows from the standard decomposition (2.10)–(2.11). □

**3.3.3** Consider now the case of singular homogeneous $\omega$. First, let us consider $\mathcal{I}_\gamma$ defined by (2.12) with $\omega = 1$ and $\psi_1,\psi_2$ replaced by $\psi_{1,\gamma},\psi_{2,\gamma}$ which are some smooth functions scaled at some point $z$ with the scaling parameter $\gamma \in (h^{1-\delta}, h^\delta)$. To have the microhyperbolicity condition sustain scaling, we replace it by (2.15). Then (3.14) implies

$$\mathcal{I}' \sim \sum_{n,m\geqslant 0} \varkappa_{nm} h^{-d+n}\gamma^{m-n+d} \tag{3.25}$$

and, obviously,

$$(2\pi)^{-d} \iint_{\{a(x,\xi)<0\}} \psi_{1,\gamma}(x)\psi_{2,\gamma}(x)\,dx\,d\xi \sim \sum_{m\geqslant 0} \varkappa'_m \gamma^{m+d}. \tag{3.26}$$

One can see easily that, in (3.25), the terms with $m = 0$ would be the same for the operator $A^0_z = a_0(z, hD)$, where $a_0(x,\xi)$ is the principal symbol of $A$; this $z$ is not necessarily the original one, but the distance between them should not exceed $c\gamma$; similarly, in (3.26), the term with $m = 0$ coincides with the left-hand expression with $a(x,\xi)$ replaced by $a(z,\xi)$.

What is more, under the condition (2.15), the integration with respect to $x$ is not needed, so all these results would hold (without factor $\gamma^d$ in the decomposition and estimates) without it; thus one can take $z=x$ (or $y$, does not matter).

Thus, we arrive at

**Proposition 3.5.** *Let $\mathcal{I}'$ be defined by* (2.12) *with $\omega=1$ and $\psi_1,\psi_2$ replaced by $\psi_{1,\gamma},\psi_{2,\gamma}$ which are the same smooth functions scaled at some point $z$ with the parameter $\gamma\in(h^{1-\delta},h^{\delta})$. Let $\mathcal{I}^{0\prime}$ be defined the same way, but with $U(t)$ replaced by $U^0(t)=e^{ih^{-1}tA^0}$, where $A^0=a(z,hD)$ and later $z$ is set to $x$. Then $\mathcal{I}'-\mathcal{I}^{0\prime}_m=O(h^{1-d}\gamma^d)$*

Now, we can calculate $\mathcal{I}$ in the scalar case:

**Proposition 3.6.** *In the frames of Proposition 3.5, as $\omega$ satisfies* (2.7) *and $\kappa>0$ $\mathcal{I}-\mathcal{I}^0=O(h^{1-d-\kappa})$, where $\mathcal{I}^0$ is defined for constant-coefficient operator obtained by freezing coefficients of $A$ at point $x$ (or $y$, does not matter).*

*Proof.* Consider three zones: $\{|x-y|\gtrsim\gamma_1\}$ with $\gamma_1\asymp h^{\delta}$, $\{\gamma\lesssim|x-y|\lesssim\gamma_1\}$ with $\gamma_0\asymp h^{1-\delta}$, and $\{|x-y|\lesssim\gamma\}$. Then the contribution of the first zone to the reminder for $\mathcal{I}$ and $\mathcal{I}^0$ does not exceed $Ch^{1-d}\gamma_1^{-1-\kappa}=O(h^{1-d-\kappa})$ (while the main parts are 0); in virtue of Proposition 3.5 and decomposition of Subsect. 2.2, the contribution of the second zone to $\mathcal{I}-\mathcal{I}^0$ does not exceed $O(h^{1-d}\gamma^{-\kappa})=o(h^{1-d-\kappa})$.

In the third zone, one can apply the method of successive approximations resulting in

$$\mathcal{I}-\mathcal{I}^0\sim h^{-d}\sum_{m+n+k\geqslant 1}\varkappa''_{mnk}h^{-d+n-m+k-\kappa}\gamma^{2m-n}.$$

However, since a final answer does not depend on $\gamma$, only the terms with $2m=n$ are posed to survive just resulting in $\bigl(\varkappa+o(1)\bigr)h^{-d+1-\kappa}$. □

Summarizing the results of Sect. 2, Proposition 3.6, and formulas (3.5)–(3.8), we arrive at

**Theorem 3.7.** *Let $A$ be a scalar operator satisfying the conditions* (1.9) *and* (2.15). *Then*

$$I=\int\mathcal{J}(x)\psi_1(x)\psi_2(x)\,dx+O(h^{1-d-\kappa}),\tag{3.27}$$

*where*

$$\begin{aligned}\mathcal{J}(x)&=2(2\pi h)^{-2d}\iiint E(x,\xi,0)\Omega(x,z)E(x,\eta,0)e^{ih^{-1}\langle z,\xi-\eta\rangle}\,dzd\xi d\eta\\&=2(2\pi h)^{-2d}\iiint_{\{a(x,\xi)<0,\ a(x,\eta)<0\}}\Omega(x,z)e^{ih^{-1}\langle z,\xi-\eta\rangle}\,dzd\xi d\eta\\&=G(x)h^{-d-\kappa},\end{aligned}\tag{3.28}$$

*with*

$$G(x) = \iint E(x,\xi,0)\widehat{\Omega}(x,\xi-\eta)E(x,\eta,0)\,d\xi d\eta$$
$$= \iint_{\{a(x,\xi)<0,\ a(x,\eta)<0\}} \widehat{\Omega}(x,\xi-\eta)\,d\xi d\eta, \tag{3.29}$$

*and* $\widehat{\Omega}$ *is defined by* (3.8).

**Remark 3.8.** (i) Alternatively, one can prove this theorem using oscillatory integral representation of $u(x,y,t)$ as $|t| \leqslant T = \varepsilon$.

(ii) Alternatively, one can replace one or both copies of $x$ in $E(x,.,.)$ or in $a(x,0)$ by $y$.

**Remark 3.9.** We refer to formulas (3.27)–(3.29), (3.8) as to the *standard Weyl expression* even in the matrix case. However, in this case, the third parts of (3.27),(3.28) should be skipped.

## *3.4 Schrödinger operator*

Now, my goal is to weaken and eventually to get rid off the microhyperbolicity condition for scalar operators. I start from the Schrödinger operator.

For the Schrödinger operator the condition of microhyperbolicity (2.15) means that

$$V \geqslant \varepsilon_0. \tag{3.30}$$

If this condition is violated, let us introduce scaling functions $\rho(x)$, $\gamma(x)$ in the usual way $\gamma = \varepsilon|V|$ and $\rho = \gamma^{1/2}$.

Then, the contribution of $B(\overline{x},\gamma(\overline{x}))^2$ to the remainder does not exceed

$$C(h/\rho\gamma)^{1-d-\kappa}\gamma^{-\kappa} \asymp Ch^{1-d-\kappa}\rho^{d-1-\kappa}\gamma^{d-1} \tag{3.31}$$

with $\rho = \rho(\overline{x})$ and $\gamma = \gamma(\overline{x})$ and then the contribution of the zone

$$\bigl\{(x,y) : |x-y| \leqslant \varepsilon\gamma(x)\bigr\} \tag{3.32}$$

(where, automatically, $\gamma(x) \asymp \gamma(y)$) to the remainder does not exceed

$$Ch^{1-d-\kappa}\int \rho^{d-1+\kappa}\gamma^{-1}\,dx \tag{3.33}$$

and with $\rho = \gamma^{1/2}$ here it becomes

$$Ch^{1-d-\kappa}\int \gamma^{(d-3+\kappa)/2}\,dx; \tag{3.34}$$

obviously, it is $O(h^{1-d-\kappa})$ provided that *either* $d+\kappa \geqslant 3$ *or*

$$|V| + |\nabla V| \geqslant \varepsilon_0. \tag{3.35}$$

and $d + \kappa > 1$ (which is surely the case).

**Remark 3.10.** (i) Note that (3.35) is the microhyperbolicity condition (2.4).

(ii) Actually, one should take $\rho\gamma \geqslant Ch$ and thus to add $Ch^{1/3}$ and $Ch^{2/3}$ to $\rho$,$\gamma$ respectively (but it does not affect our conclusion due to the standard fact that if $\rho\gamma \asymp h$, then $h_{\text{eff}} \asymp 1$) and the condition (3.35) is not needed.

Consider now the complement of the zone (3.32). Let us redefine there $\gamma(x)$ as $\gamma(x, y) = \frac{1}{2}|x - y|$ and, in this zone, the condition (3.35) is not needed as one can see easily after rescaling $B(x, \gamma(x, y))$ to $B(0, 1)$ due to Proposition 2.5.

Therefore, as $\gamma \geqslant \gamma(x)$ the contribution of $B(x, \gamma)^2 \setminus \{\text{zone (3.32)}\}$ to the remainder does not exceed the same expression (3.31) with $\rho = \gamma^{1/2}$. Then the contribution of the complement of the zone (3.32) to the remainder does not exceed

$$Ch^{1-d-\kappa} \iint_{\{|x-y| \geqslant \varepsilon \max(\gamma(x), \gamma(y))\}} |x - y|^{(d-1+\kappa)/2-1-d}\, dx\, dy. \tag{3.36}$$

One can see easily that the expression (3.36) is $O(h^{1-d-\kappa})$ as $d + \kappa > 3$ (so this case is already covered).

Further, the expression (3.36) does not exceed the expression (3.34) with $\gamma = \gamma(x)$ and the expression

$$Ch^{1-d-\kappa} \int (|\log \gamma(x)| + 1)\, dx \tag{3.37}$$

as $d + \kappa < 3$ and $d + \kappa = 3$ respectively and both these expressions are $O(h^{1-d-\kappa})$ under the condition (3.35).

Again, we get $O(h^{1-d-\kappa})$ provided that either $d + \kappa > 3$ or the condition (3.35) is fulfilled. So, we arrive at

**Proposition 3.11.** *Consider the Schrödinger operator. Let either $d + \kappa > 3$ or the condition* (3.35) *be fulfilled. Then the standard Weyl asymptotics* (3.27)–(3.29)*,* (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

This completely covers the case $d \geqslant 3$. Furthermore, after Proposition 3.11 is proved, we can introduce scaling functions $\gamma = \rho = \varepsilon(|V| + |\nabla V|^2)^{1/2} + Ch^{1/2}$ and then, applying the same arguments, we arrive at

**Proposition 3.12.** *Consider the Schrödinger operator. Let either $d + \kappa > 2$ or the condition*

$$|V| + |\nabla V| + |\nabla^2 V| \geqslant \varepsilon \tag{3.38}$$

*be fulfilled. Then the standard Weyl asymptotics* (3.27)–(3.29)*,* (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

This completely covers the case $d = 2$. As $d = 1$ we get the required remainder estimate under the condition (3.38).

Now, combining this with the arguments of the proof of Theorem 4.4.9 of [6], we get[5]

**Proposition 3.13.** *Consider the Schrödinger operator with* $d = 1$, $\kappa > 0$. *Then the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate* $O(h^{1-d-\kappa})$.

**Remark 3.14.** Actually, all above results hold as $\kappa = 0$ as well with the singular exception of $d = 1$ when the remainder estimate $O(1)$ is recovered under the condition

$$\sum_{|\beta|\leqslant K} |\nabla_x^\beta V| \geqslant \varepsilon; \tag{3.39}$$

without it the remainder estimate is $O(h^{-\delta})$ with arbitrarily small $\delta > 0$.

## *3.5 Scalar case. II*

**3.5.1** Let us consider general scalar operators.

**Remark 3.15.** (i) Actually, instead of the condition (1.9), one can make a cut-off with respect to $\xi$ replacing functions $\psi_j(x)$ by pseudodifferential operators $\psi_j(x, hD)$ with smooth compactly supported symbols;

(ii) Alternatively, we can replace $E(0)$ by $E(\tau, \tau') = E(\tau) - E(\tau')$ with conditions satisfied for $a - \tau$ and $a - \tau'$ instead of $a$.

(iii) Alternatively, we can replace $E(0)$ by

$$E'(\tau) = \int\limits_{\mathbb{R}} E(0, \tau')\varphi(\tau')\, d\tau' \tag{3.40}$$

with smooth function $\varphi$ s.t. $\int\limits_{\mathbb{R}} \varphi(\tau')\, d\tau' = 1$.

In all these cases, obvious modifications of the final formulas are needed.

Now, we can introduce scaling functions

$$\gamma(x, \xi) = \varepsilon\bigl(|\nabla_\xi a|^2 + |a|\bigr) + Ch^{2/3}, \qquad \rho(x, \xi) = \gamma^{1/2}(x, \xi) \tag{3.41}$$

and repeat arguments of the previous subsection; then the expression (3.33) will be replaced by $Ch^{1-d-\kappa}M$ with

$$M = \int \rho^{\kappa-1}\gamma^{-1}\, dxd\xi \asymp \int \bigl(|\nabla_\xi a|^2 + |a|\bigr)^{(\kappa-3)/2}\, dxd\xi \tag{3.42}$$

[5] I am leaving easy details to the reader; see also the proof of Theorem 3.19.

(in the zone $\{\rho\gamma \geqslant Ch\}$). Therefore, we arrive at the remainder estimate $O(h^{1-d-\kappa})$ provided that $M = O(1)$ as *now the integral in $M$ is taken over* $B(0,1)$.

This is definitely the case as $\kappa \geqslant 3$. Assume now that the microhyperbolicity condition (2.4) is fulfilled. Then $M = O(1)$ as $\kappa > 1$; otherwise, this condition becomes

$$\int\limits_{\Sigma} |\nabla_\xi a|^{\kappa-1}\, d\mu < \infty \quad \text{as } 0<\kappa<1, \qquad \int\limits_{\Sigma} |\log|\nabla_\xi a||\, d\mu < \infty \tag{3.43}$$

with $\Sigma = \{a(x,\xi)=0\}$ and $d\mu = dxd\xi : da$ measure on $\Sigma$.

Thus, we arrive at the following generalization of Proposition 3.11:

**Proposition 3.16.** *Let $A$ be a scalar operator satisfying the condition* (1.9). *Assume that the uniform version of the condition*[6]

$$a = \nabla_\xi a = 0 \implies \operatorname{rank} \mathrm{Hess}_{\xi\xi} a \geqslant r \tag{3.44$_r$}$$

*is fulfilled. Then*

(i) *As $r+\kappa > 3$ the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$;*

(ii) *Under the condition* (2.4), *as $r+\kappa>1$ the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

*Proof.* In contrast to standard asymptotics, we need to consider not points $(x,\xi)$, but pairs $(x,\xi;y,\eta)$ and the pure standard arguments work in the zones

$$\bigl\{(x,\xi;y,\eta) : |x-y| \leqslant \varepsilon\gamma(x,\xi), |\xi-\eta| \leqslant \varepsilon\rho(x,\xi)\bigr\} \tag{3.45}$$

where also $\gamma(y,\eta) \asymp \gamma(x,\xi)$ and $\rho(y,\eta) \asymp \rho(x,\xi)$. Analysis in the complimentary zone I postpone until the proof of Theorem 3.19, where it will be done in more general settings. □

Now, introducing scaling functions

$$\gamma(x,\xi) = \varepsilon\bigl(|\nabla_{x,\xi} a|^2 + |a|\bigr)^{1/2} + Ch^{1/2}, \qquad \rho(x,\xi) = \gamma(x,\xi) \tag{3.46}$$

and repeating the same arguments we arrive at the following generalization of Proposition 3.12:

**Proposition 3.17.** *Let $A$ be a scalar operator satisfying the condition* (1.9). *Assume that the uniform version of the condition* (3.44)$_r$ *is fulfilled. Then as $r+\kappa>2$, the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

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[6] I.e., $|a| + |\nabla_\xi a| \leqslant \varepsilon$ implies that $\mathrm{Hess}_{\xi\xi} a$ has $r$ eigenvalues which absolute values are greater than $\varepsilon$.

Again, combining this with the arguments of Theorem 4.4.9 of [6] we arrive at the following generalization of Proposition 3.12.

**Proposition 3.18.** *Let $A$ be a scalar operator satisfying the conditions* (1.9) *and* $(3.44)_1$*, and let $\kappa > 0$. Then the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

**3.5.2** Now, we can prove our main result for scalar operators:

**Theorem 3.19.** *Consider scalar operator. Let the conditions* (1.9) *and*

$$\sum_{0\leqslant k\leqslant n} |\nabla_\xi^k a| \geqslant \varepsilon_0 \tag{$3.47_n$}$$

*with some $n$ be fulfilled. Let $\omega$ satisfy* (2.7) *and $\kappa > 0$. Then the standard Weyl asymptotics* (3.27)–(3.29), (3.8) *holds with the remainder estimate $O(h^{1-d-\kappa})$.*

*Proof. Part I.* In this part of the proof, we consider at each step only the zone (3.45), where $\gamma$ will be defined in different ways later. The treatment of the complementary zone will be described in Part II.

So, we proved the statement of the theorem under the condition $(3.44)_1$ which is equivalent to $(3.47)_2$.

Let us apply an induction with respect to $n$. *Assume that, under the condition $(3.47)_n$, the required estimate is proved.*

In the general case (without the condition $(3.47)_n$), we can introduce scaling functions in the manner similar to (3.41):

$$\begin{aligned} &\gamma(x,\xi) = \varepsilon\Bigl(\sum_{0\leqslant k\leqslant n} |\nabla_\xi^k a|^{N/(n-k+1)}\Bigr)^{(n+1)/N} + Ch^{(n+1)/(n+2)},\\ &\rho(x,\xi) = \gamma^{1/(n+1)}(x,\xi) \end{aligned} \tag{$3.48_n$}$$

with $N = (n+1)!$.

Therefore, under assumption of induction, we get again the remainder estimate $Ch^{1-d+\kappa}M$ with $M$ given by (3.42), where this time the right-hand expression becomes

$$M = \int \gamma^{(\kappa-n-2)/(n+1)}\, dxd\xi; \tag{$3.49_n$}$$

under the condition (2.4) this expression becomes

$$M \asymp \int_\Sigma \gamma^{(\kappa-1)/(n+1)}\, d\mu \asymp \int_\Sigma \Bigl(\sum_{1\leqslant k\leqslant n} |\nabla_\xi^k a|^{1/(n-k+1)}\Bigr)^{\kappa-1} d\mu \tag{$3.50_n$}$$

which is $O(1)$ under assumption $|\nabla_\xi^{n+1} a| \geqslant \varepsilon_0$ (as lower order derivatives with respect to $\xi$ are close to 0). This is exactly the condition $(3.47)_{n+1}$.

*So, now we have a proper estimate under the condition* $(3.47)_{n+1}$ *instead of* $(3.47)_n$*, but now we also need the condition* (2.4).

Without the condition (2.4), we would need something different; for example, ignoring the integration with respect to $x$, one should assume that $\operatorname{rank}(\nabla_\xi^{n+1} a) + \kappa > n+2$, where the rank of multilinear symmetric $m$-form $G$ is $d - \dim \operatorname{Ker} G$; $\operatorname{Ker} G = \{x : G(x, x^2, \dots, x^m) = 0\ \forall x^2, \dots, x^m\}$. This is rather unusable.

Instead, I want to weaken the condition (2.4), replacing it by

$$\sum_{2\leqslant j\leqslant n+1,\ l:m+j:(n+1)\leqslant 1} |\nabla_x^l \nabla_\xi^j| \geqslant \varepsilon_0 \tag{$(3.51)_{n+1,m}$}$$

for some $m > 0$ which is not necessarily an integer. Obviously in our assumptions (2.4) coincides with $(3.51)_{n+1,1}$.

Let us run a kind of nested induction. So, *let us assume that, under the conditions* $(3.47)_{n+1}$ *and* $(3.51)_{n+1,m}$*, the remainder estimate* $O(h^{1-d-\kappa})$ *is proved.*

Now, we can go to something similar (3.46):

$$\begin{aligned}
&\gamma(x,\xi) = \varepsilon\Big(\sum_{k,l:k:n+l:m\leqslant 1} |\nabla_\xi^k \nabla_x^l a|^{N s_{kl}}\Big)^{1/N} + \overline{\gamma},\\
&\overline{\gamma} = Ch^{(n+1)/(m+n+2)}, \quad \rho(x,\xi) = \gamma^{(m+1)/(n+1)}(x,\xi),\\
&s_{kl} = \frac{n+1}{(m+1)(n+1) - (m+1)k - (n+1)l}.
\end{aligned} \tag{$(3.52)_{n,m}$}$$

Then we recover the remainder estimate $Ch^{1-d-\kappa} M$ with $M$ defined by (3.42) which is now

$$M \asymp \int \gamma^{-1+(m+1)(\kappa-1)/(n+1)}\, dxd\xi \asymp \int \rho^{-(n+1)/(m+1)+(\kappa-1)}\, dxd\xi. \tag{$(3.53)_{nm}$}$$

Under the condition $(3.47)_{n+1}$, we can assume without any loss of generality that

$$a(x,\xi) = \sum_{0\leqslant j\leqslant n+1} b_j(x,\xi')\xi_1^{n+1-j}, \qquad b_0 = 1, \quad b_1 = 0; \tag{3.54}$$

we can always reach it by change of coordinates and multiplication of $A$ by an appropriate positive pseudo-differential factor. Then

$$\begin{aligned}
&\rho \asymp |\xi_1| + \widetilde{\rho}, \quad \widetilde{\rho} = \widetilde{\gamma}(x,\xi')^{(m+1)/(n+1)},\\
&\widetilde{\gamma} = \sum_{j,k,l:(k+j):n+(l:m)\leqslant 1} |\nabla_{\xi'}^k \nabla_x^l b_j|^{s_{(k+j)l}} + \overline{\gamma}.
\end{aligned} \tag{3.55}$$

Then

$$M \asymp \int \widetilde{\rho}^{-(n+1)/(m+1)+\kappa}\,dxd\xi' \asymp \int \widetilde{\gamma}^{-1+(m+1)\kappa/(n+1)}\,dxd\xi' \qquad (3.56)_{nm}$$

(with an extra logarithmic factor as the power is 0). Then $M = O(1)$ as

$$(m+1)\kappa/(n+1) > 1. \qquad (3.57)$$

Moreover, $M = O(1)$ provided that there exists $(j,k,l)$ with $|\nabla_{\xi'}^k \nabla_x^l b_j| \geqslant \varepsilon_0$ and *either* $k \geqslant 1$, $(k+j-1) : n+l : m \leqslant 1$, $s_{k+j-1,l} < 1$ *or* $l \geqslant 1$, $(k+j) : n+(l-1) : m \leqslant 1$, $s_{k+j,l-1} < 1$.

Therefore, one can derive easily

> If the remainder estimate $O(h^{1-d-\kappa})$ holds under the condition $(3.51)_{n+1,m'}$ for every $m' < m$, then it also holds under the condition $(3.51)_{n+1,m}$. (3.58)

On the other hand, there exists a discrete set $\{m_\nu\}_{\nu=1,2,\dots}$ with $m_1 < m_2 < \dots$ such that if the condition $(3.51)_{n+1,m}$ is fulfilled for $m = m_\nu$, then it is fulfilled for all $m \in (m_\nu, m_{\nu+1})$ as well.

This justifies induction with respect to $m$ running this set and therefore, the remainder estimate $O(h^{1-d-\kappa})$ holds under the condition $(3.51)_{n+1,m}$ no matter how large $m$ is. However, if $m$ is large enough, the condition (3.57) is fulfilled and we do not need the condition $(3.51)_{n+1,m}$ anymore.

This concludes induction with respect to $n$. □

*Proof. Part II.* However, in contrast to standard asymptotics, we need to consider not points $(x,\xi)$, but pairs $(x,\xi;y,\eta)$ and the pure standard arguments work in the zone (3.45).

It follows from the standard theory that if $Qx$ and $Q_y$ have symbols supported in $\varepsilon(\rho_x,\gamma_x)$- and $\varepsilon(\rho_y,\gamma_y)$-vicinities of $(x,\xi)$ and $(y,\eta)$ respectively, then

$$\|Q_x E Q_y\|_1 \leqslant Ch^{-d}(\rho_x\gamma_x)^{d/2}(\rho_y\gamma_y)^{d/2}; \qquad (3.59)$$

moreover, if either $|x-y| \geqslant \varepsilon_0\gamma_x$ or $|\xi-\eta| \geqslant \varepsilon_0\rho_x$, then

$$\|Q_x E Q_y\|_1 \leqslant Ch^{1-d}(\rho_x\gamma_x)^{d/2-1}(\rho_y\gamma_y)^{d/2}. \qquad (3.60)$$

Surely, the same will be true with $(x,\xi)$ and $(y,\eta)$ permuted.

Then the contribution of such a pair to the error estimate does not exceed

$$Ch^{1-d}(\rho_x\gamma_x)^{d/2-1}(\rho_y\gamma_y)^{d/2}|x-y|^{-\kappa} \qquad (3.61)$$

if $|x-y| \geqslant \varepsilon_0\gamma_x$.

Otherwise, the contribution of the pair $\psi_x Q_x$ and $Q_y$ to the error estimate does not exceed $Ch^{1-d}(\rho_x\gamma_x)^{d/2-1}(\rho_y\gamma_y)^{d/2}\gamma^{-\kappa}$, where $\psi_1$, $(1-\psi_x)$ are supported in $\{|x-y|\geqslant\gamma\}$ and $\{|x-y|\leqslant 2\gamma\}$ and $\gamma\geqslant h\rho_x^{-1}$.

Furthermore, since

$$|Q_x E Q_y| \leqslant Ch^{1-d}\rho_x^{d/2-1}\gamma_x^{-1}\rho_y^{d/2} \tag{3.62}$$

due to the standard arguments, the contribution of the pair $(I-\psi_x)Q_x$ and $Q_y$ to the error does not exceed $Ch^{2-2d}\rho_x^{d-2}\rho_y^d\gamma_x^{-2}\gamma_y^d\gamma^{d-\kappa}$ Plugging $\gamma=h\rho_x^{-1}$, we estimate the contribution of the pair $Q_x$, $Q_y$ by

$$Ch^{1-d-\kappa}(\rho_x\gamma_x)^{d/2-1}(\rho_y\gamma_y)^{d/2}\rho_x^{\kappa}+Ch^{2-d-\kappa}\rho_x^{-2+\kappa}\gamma_x^{-2}\rho_y^d\gamma_y^d \tag{3.63}$$

which is larger than (3.61).

In these estimates, we do not need nondegeneracy condition and therefore, as $(y,\eta)$ and $(x,\xi)$ are given, we can take

$$\rho_x=\rho_y=|x-y|^{\sigma}+|\xi-\eta|, \gamma_x=\gamma_y|x-y|+|\xi-\eta|^{1/\sigma}, \tag{3.64}$$

where $\rho=\gamma^{\sigma}$ on the corresponding step of our analysis. Then as $(z,\zeta)$ are fixed contribution of $\{|x-z|\leqslant\gamma, |y-z|\leqslant\gamma, |\xi-\zeta|\leqslant\rho, |\eta-\zeta|\leqslant\rho, |x-y|+|\xi-\eta|^{1/\sigma}\geqslant\varepsilon\gamma\}$ to the error does not exceed this expression

$$Ch^{1-d-\kappa}(\rho\gamma)^{d-1}\rho^{\kappa}+Ch^{2-d-\kappa}(\rho\gamma)^{d-2}\rho^{\kappa}, \tag{3.65}$$

where the second term is less than the first one.

Then the total contribution of the zone in question to the error does not exceed

$$Ch^{1-d}\iiint \gamma^{\sigma\kappa-\sigma-1}\, dyd\eta\, \gamma^{-1}d\gamma \tag{3.66}$$

where equation is taken over $\{\gamma\geqslant\gamma_x\}$ and the integral in question is equivalent to $Mh^{1-d}$, where $M=1$ as $\sigma(\kappa-1)>1$,

$$M=\iiint |\log\gamma(y,\eta)|\, dyd\eta \tag{3.67}$$

as $\sigma(\kappa-1)=1$ and, due to $(3.47)_n$, $M\asymp 1$ as well,

$$M=\iiint \gamma(y,\eta)^{\sigma\kappa-\sigma-1}\, dyd\eta \tag{3.68}$$

as $\sigma(\kappa-1)<1$, and, on each step of the induction, we already proved that $M\asymp 1$. □

### *3.6 General microhyperbolic case. II*

Let us consider a matrix operator. Let $\lambda_j(x,\xi)$ be eigenvalues of its principal part. Then $|\nabla_{x,\xi}\lambda_j|\leqslant c$ and microhyperbolicity with respect to $\ell$ means that

$$|\lambda_j(x,\xi)|\leqslant \varepsilon_0 \implies (\ell\lambda_j)(x,\xi)\geqslant \varepsilon_0 \qquad \forall j. \tag{3.69}$$

Let us consider the zone

$$\mathcal{U}_j=\bigl\{(x,\xi): |\lambda_j|\lesssim \min_{k\ne j}|\lambda_k|\bigr\}, \tag{3.70}$$

and let us define here

$$\gamma \stackrel{\rm def}{=} \min_{k\ne j}|\lambda_k|+\frac{1}{2}\overline{\gamma} \tag{3.71}$$

and $\rho=\gamma$. Consider the zone

$$\bigl\{\gamma\geqslant |x-y|+|\xi-\eta|+\overline{\gamma}\bigr\} \tag{3.72}$$

and rescale $x\mapsto x/\gamma$, $\xi\mapsto\xi/\gamma$, $\lambda_k\mapsto\lambda_k/\gamma$, $h\mapsto h/\gamma^2$ preserving the microhyperbolicity condition (2.15) and simultaneously making an operator with $|\lambda_k|\geqslant 1$ for $k\ne j$ and therefore, the analysis of this operator is not different from the scalar one. Unfortunately, we cannot use the nondegeneracy conditions of Subsects. 3.4–3.5 which would not survive this, but the microhyperbolicity condition survives and *we assume that* (2.15) *is fulfilled*.

Then as the main part of the asymptotics is given by the standard Weyl expression (3.27)–(3.29), the contribution of the zone (3.72) (intersected with $\{\gamma\geqslant C_0\overline{\gamma}\}$) to the remainder does not exceed

$$\begin{aligned} R_j &= \int\limits_{\varSigma_j\cap\{\gamma\geqslant C\overline{\gamma}\}} C\bigl(h\gamma^{-2}\bigr)^{1-d-\kappa}\gamma^{-\kappa-2d}\,d\wp_j \\ &\asymp Ch^{1-d-\kappa}\int\limits_{\varSigma_j\cap\{\gamma\geqslant\overline{\gamma}\}}\gamma^{-2+\kappa}\,d\wp_j \end{aligned} \tag{3.73}$$

with $\varSigma_j=\{(x,\xi):\lambda_j=0\}$ and $d\wp_i=dxd\xi:d\lambda_j$ density on it.

Let us fix $\overline{\gamma}=Ch^{1/2}$. Then, in the complementary zone, $\cup_{k\ne j}\{|\lambda_j|+|\lambda_k|\leqslant C\overline{\gamma}\}$ one needs just to make a rescaling $x\mapsto x/\overline{\gamma}$, $\xi\mapsto\xi/\overline{\gamma}$ which sends $h$ to 1 and no microhyperbolicity condition would be needed and the contribution of this zone would not exceed

$$R'_{jk}=Ch^{-d-\kappa/2}\mathrm{meas}\bigl\{|\lambda_j|+|\lambda_k|\leqslant Ch^{1/2}\bigr\}. \tag{3.74}$$

So, the total contribution of the zone $\cup_j \mathcal{U}_j$ to the remainder is given by $\sum_j R_j + \sum_{j,k:j\neq k} R'_{jk}$.

Assuming that

$$\wp_j\bigl(\Sigma_j : |\lambda_k| \leqslant t\bigr) + t^{-1}\mathrm{meas}\{|\lambda_j| + |\lambda_k| \leqslant t\} = O(t^r), \tag{3.75}$$

we get, under the additional assumption $r+\kappa>2$ (which is always fulfilled as $r\geqslant 2$), that $R_j = O(h^{1-d-\kappa})$ while $R'_{jk} = O(h^q)$ with

$$q = -d - \frac{1}{2}\kappa + \frac{r}{2}, \tag{3.76}$$

which is $O(h^{1-d-\kappa})$ as well.

On the other hand, as $r+\kappa<2$ we get that $R_j = O(h^q)$, $R'_{jk} = O(h^q)$ with $q$ given by (3.77).

Finally, as $r+\kappa=2$ we get $R_j = O(h^{1-d-\kappa}|\log h|)$ and $R_{jk} = O(h^{1-d-\kappa})$.

Assume temporarily that no more than two eigenvalues can be close to 0 simultaneously. Then we are already done since, in the zone complimentary to (3.72), we redefine $\gamma = \varepsilon(|x-y| + |\xi-\eta|)$ and apply the same rescaling as before and one does not need microhyperbolicity condition.

Let us apply the induction by $m$ assuming that no more than $m$ eigenvalues can be close to 0 simultaneously. Then we can define on each step

$$\gamma(x,\xi) = \varepsilon \max_{J:\#J=m} \min_{k\notin J} |\lambda_k(x,\xi)| + \overline{\gamma} \tag{3.77}$$

and repeat all above arguments. We arrive at

**Theorem 3.20.** *Let the conditions* (1.9), (1.7), (2.15), *and* (3.75) *be fulfilled. Then the standard Weyl asymptotics* (3.27)–(3.29) *holds with the remainder estimate*

(i) *which is* $O(h^{1-d-(m-1)\kappa})$ *as* $r+\kappa>2$;

(ii) *which is* $O(h^q)$ *with* $q$ *defined by* (3.76) *as* $r+q\kappa<2$ *and* $O(h^{1-d-\kappa}|\log h|)$ *as* $r+\kappa=2$.

**Remark 3.21.** The condition (3.75) is fulfilled provided that $\Lambda_{jk} = \{\lambda_j = \lambda_k = 0\}$ are smooth manifolds of codimension $r$ and $|\lambda_j| \asymp |\lambda_k| \asymp \mathrm{dist}((x,\xi),\Lambda_{jk})$ in its vicinity; this assumption should be fulfilled for all $j\neq k$.

## References

1. Bronstein, M., Ivrii, V.: Sharp Spectral Asymptotics for Operators with Irregular coefficients. Pushing the limits. Commun. Partial Differ. Equ. **28**, no. 1-2, 99-123 (2003)

2. Ivrii, V.: Semiclassical asymptotics for exchange energy. Séminaire sur les Équations aux Dérivées Partielles, 1993–1994, Exp. No. XX, 12 pp., Ecole Polytech., Palaiseau (1994)
3. Ivrii, V.: Asymptotics of the ground state energy of heavy molecules in the strong magnetic field. I. Russian J. Math. Phys. **4**, no 1, 29-74 (1996)
4. Ivrii, V.: Heavy molecules in the strong magnetic field. Russian J. Math. Phys. **4**, no 4, 449-456 (1996)
5. Ivrii, V.: Asymptotics of the ground state energy of heavy molecules in the strong magnetic field. II. Russian J. Math. Phys. **5**, no 3, 321-354 (1997)
6. Ivrii, V.: Microlocal Analysis and Precise Spectral Asymptotics. Springer-Verlag (1998)
7. Ivrii, V., Sigal, I.M.: Asymptotics of the ground state energies of large Coulomb systems. Ann. Math. **138**, 243–335 (1993)

# Linear Hyperbolic and Petrowski Type PDEs with Continuous Boundary Control → Boundary Observation Open Loop Map: Implication on Nonlinear Boundary Stabilization with Optimal Decay Rates

Irena Lasiecka and Roberto Triggiani

**Abstract** Uniform stabilization with nonlinear boundary feedback is asserted for classes of hyperbolic and Petrowski type multidimensional partial differential equations with variable coefficients (in space), as a consequence of the continuity (boundedness) of the corresponding purely boundary control → boundary observation open-loop map of dissipative character, of interest in its own right. The interior is assumed inaccessible. There are explicit hyperbolic/Petrowski type dynamical PDE classes where such a property holds and classes where it fails. When available, it has a number of attractive and unexpected consequences. In particular, when accompanied by exact controllability of the corresponding open-loop linear model, it implies uniform stabilization with optimal decay rates—when a nonlinear function of the boundary observation closes up the loop, to generate the corresponding boundary feedback dissipative problem.

## 1 Open-Loop and Closed-Loop Abstract Setting for Hyperbolic/Petrowski Type PDEs with Boundary Control

The common theme across the three Springer volumes commemorating the centenary of Sergey Sobolev is *Sobolev Spaces in Mathematics.* The present volume is then more specifically focused on *Applications in Mathematical Physics.* What an ideal and most fitting carrier or medium, then, to pay tribute to the intent of the volumes than selecting partial differential equations!

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series.*
doi: 10.1007/978-0-387-85652-0, 

Sobolev spaces are the established and universal language of partial differential equations: they permeate the settings within which partial differential equations are studied and analyzed. This is, in particular, the case of the modern control theory of partial differential equations. Here, desired behaviors of the solutions of dynamical systems are actively synthesized by an appropriate choice of input or control functions, in order to achieve the pre-determined goals. In the case of partial differential equations, control functions are typically, and most challengingly, sought to act on the boundary of the bounded spatial domains, whereby the precise link between the Sobolev regularity of boundary control function and the Sobolev interior regularity, or Sobolev boundary regularity of a suitable trace, of the solution becomes critical. The present paper fits in the aforementioned framework.

### *1.1 A key open-loop boundary control–boundary observation map: orientation*

In this paper, we focus on a suitable open-loop boundary control → boundary observation regularity property of dissipative character for *linear* (second order and first order) hyperbolic and Petrowski type multidimensional PDEs. Besides being of interest in itself, this property—when it holds —opens the door to a variety of optimal control/min-max purely boundary problems; and, moreover, it has unexpected links and consequences. This will be elaborated below with more details. Thus, in this paper, we revisit a boundary → boundary regularity issue already studied in our past efforts [48, 49]. Now we complement these references by providing new insight, new positive and negative examples, new connections. Moreover, we extend these references' setting (by considering equations with variable coefficients in space in the principal part), as well as their scope (by encompassing a markedly larger class of nonlinear boundary feedbacks with no *a priori* assumptions near the origin). Indeed, here we choose to pursue the link between the validity of this open-loop boundary control → boundary observation regularity property of the original *linear* problem and its consequences on the uniform stabilization of a corresponding closed-loop problem with *nonlinear* dissipative boundary feedback. We do so, both at the abstract level, as well as for several "concrete" and explicit classes of hyperbolic and Petrowski type multidimensional PDEs, with variable coefficients (in the space variable). Within such dynamics, there are explicit PDE classes—originally identified in [48, 49], and further expanded here—where such a property holds and classes where it fails. As indicated above, when such (strong and desirable) boundary control → boundary observation regularity property is established, a number of attractive and unexpected consequences follow. First, it permits the setting, and consequent study, of the optimal control, or min-max game theory problem with a purely boundary control/boundary observation cost functional.

Second, it implies (but generally it is not implied by) a desirable interior regularity result, from the boundary control to the solution in the interior, of the type that has been shown for first order hyperbolic systems [15, 71], and for second order hyperbolic equations with Dirichlet boundary control [26, 27, 22], [31]–[38], [41]–[44], [63, 46, 87], and their numerous successors. Third, it provides an explicit link between two open-loop controls—the one for the original conservative system and the one for the dissipative system—that steer the same initial condition to rest, along their respective dynamics.

Finally, when accompanied by exact controllability (equivalently, continuous observability) of the corresponding *linear* model, it implies uniform stabilization with optimal decay rates—according to the strategy laid out in [24] in the case of wave equations and exported to many other dynamics [47] (shells), [51] (Schrödinger equations)—when a nonlinear function of the "open-loop dissipative" boundary observation closes up the loop, to generate a corresponding boundary feedback, closed-loop, dissipative nonlinear problem. A distinctive feature of said uniform stabilization strategy of the nonlinear boundary problem is that optimal decay rates for the energy of the closed-loop boundary nonlinear feedback system can be derived via an explicitly constructed, nonlinear, monotone, first order, separable ordinary differential equation, without any *a priori* knowledge of the behavior of the dissipation at, or near, the origin (which is the region responsible for the decay rates).

In our presentation here, we opt to emphasize, at first, the desirable continuity property of the aforementioned boundary control → boundary observation open-loop map, and to reveal its dissipative character, in the context of concrete PDE models, where it has, moreover, a physically attractive interpretation. For each class given here where such boundary → boundary linear map is bounded (continuous), we then present the ultimately sought-after nonlinear stabilization result for the corresponding closed-loop boundary dissipative problem. This is obtained by simply taking the open-loop map, applying a suitable nonlinear function to it (in particular, the identity), and closing up the associated loop.

We begin with the most challenging case: second order wave equations with variable coefficients (in the space variable). Here, we provide and put together a clean, consequential presentation of the results of [48, 49] (as they stand in references difficult to retrieve, as the publication of that journal has been discontinued for over a year due to a change of publishing policy, it is not handy to utilize [49] to correct an erroneous statement which occurred in [48], which, however, required a minor correction in the proof to reach the opposite conclusion). As already noted, for this class of wave equations under Dirichlet-boundary control, establishing the desired continuity property of the (Dirichlet-) boundary control → (Neumann-) boundary observation is a challenging task of regularity theory, which goes well beyond the results and techniques of the otherwise comprehensive treatment of regularity of [22, 26, 27]. In our proof presented in Sect. 4, it requires, in fact, an additional pseudodifferential analysis in the "elliptic sector" of the dual (or Fourier) variables. We also provide—at the end of Sect. 2—classes of PDEs where the aforementioned boundary control → boundary observation open-loop map fails to be bounded (continuous): typically, these PDE models involve the Neumann boundary control, and thus fail to satisfy the Lopatinski condition. See, however, Sect. 9, involving the Schrödinger equation with Neumann boundary control with respect to two state spaces $H^{\varepsilon}(\Omega)$, $\varepsilon > 0$, and $L_2(\Omega)$.

Throughout the paper, the PDE classes will be based on the following second order differential expression

$$\begin{aligned} \mathbb{A}w &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right) \\ \text{with } \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\geqslant a\sum_{i,j=1}^{n}\xi_i^2, \quad x\in\Omega,\ a>0, \end{aligned} \tag{1.1.0}$$

with real coefficients $a_{ij} = a_{ji}$, of class $C^2$, satisfying the uniform ellipticity condition for some positive constant $a > 0$. When supplied with suitable homogeneous boundary conditions (such as in (4.1.6)), $\mathbb{A}$ becomes a negative, self-adjoint operator. We recover $(\Delta)$ in the case of constant coefficients.

## *1.2 An historical overview on regularity, exact controllability, and uniform stabilization of hyperbolic and Petrowski type PDEs under boundary control*

At first, naturally, PDE boundary control theory for evolution equations tackled the most established of the PDE classes—parabolic PDEs—whose Hilbert space theory for mixed problems was already available in close to an optimal book-form [60, 65] since the early '70s. Next, in the early '80s, when the study of boundary control problems for (linear) PDEs began to address hyperbolic and Petrowski type systems on a multidimensional bounded domain [28, 8] (see the books [5, 39, 46] for overview), it faced at the outset an altogether new and fundamental obstacle, which was bound to hamper any progress. Namely, that an optimal, or even sharp, theory on the preliminary, foundational questions of well-posedness and global regularity (both in the interior and on the boundary, for the relevant solution traces) was generally missing in the PDEs literature of Mixed (Initial and Boundary Value) Problems for hyperbolic and Petrowski type systems [60]. Available results were often explicitly recognized as definitely non-optimal [65, Vol. 2, p. 141]

**Hard analysis energy methods.**

A happy and quite challenging exception was the optimal—both interior and boundary—regularity theory for mixed, non-symmetric, non-characteristic first order hyperbolic systems, culminated through repeated efforts in the early '70s [15, 70, 71]. Its final, full success required eventually the use of pseudodifferential energy methods (Kreiss' symmetrizer). Apart from this isolated case, mathematical knowledge of global optimal regularity theory of hyperbolic and Petrowski type mixed problems was scarce, save for some trivial one-dimensional cases. Thus, in this first incipient phase, one may say that optimal control theory [28, 8, 60] provided a forceful impetus in seeking to attain an optimal global regularity theory for these classes of mixed PDE problems. To this end, PDE (hard analysis) energy methods—both in differential and pseudodifferential form—were introduced and brought to bear on these problems. The case of second order hyperbolic equations under Dirichlet boundary control was tackled first. The resulting theory that turns out to be optimal and does not depend on the space dimension [26, 27, 45, 22, 61]. It was best achieved by the use of energy methods in differential form. The case of second order hyperbolic equations, this time under Neumann boundary control, proved far more recalcitrant and challenging (in space dimension strictly greater than one), and was conducted in a few phases. The additional degree of difficulties for this mixed PDE class stems from the fact that the Lopatinski condition is not satisfied for it. Unlike the Dirichlet boundary control, the Neumann boundary control case requires pseudodifferential analysis. Final results depend on the geometry [34, 36, 38, 45, 76].

Naturally, in investigative efforts which moved either in a parallel or in a serial mode, the conceptual and computational "tricks" that had proved successful in obtaining an optimal, or sharp, regularity theory for second order hyperbolic equations, were exported, with suitable variations and adaptations, to certain Petrowski type systems (see, for example, [31, 32, 35, 40, 41, 43, 44, 62, 64]. The lessons learned with second order equations served as a guide and a benchmark study for these other classes. To be sure, not all cases have been, to date, completely resolved. The problem of optimal regularity of some Petrowski systems with "high" boundary operators is not yet fully solved. However, a large body of optimal regularity theory has by now emerged, dealing with systems such as: Schrödinger equations; plate-like equations of both hyperbolic (Kirchhoff model) and non-hyperbolic type (Euler–Bernoulli model), etc. Subsequently, additional more complicated dynamics followed, such as: system of elasticity, Maxwell equations, dynamic shell equations, etc. A rather broad account of these issues under one cover may be found in [39, 62, 64, 45], [46, Vol. 2], etc.

**Abstract models of PDE mixed problems.**

Simultaneously, and in parallel fashion, the aforementioned investigative efforts since the mid-70's also produced "abstract models" for mixed PDE problems, subject to control either acting on the boundary of, or else as a point control, within a multidimensional bounded domain: [2, 93, 94] for parabolic problems; [80, 26, 27] for hyperbolic problems. Though, in particular, operators arising in the abstract model depend on both the specific class of PDEs and on its specific homogeneous and nonhomogeneous boundary conditions, one cardinal point reached in this line of investigation was the following discovery: that most of them—but by no means all [7, 23, 86]—are encompassed and captured by the abstract model:

$$\dot{y} = Ay + Bu, \quad \text{in} \quad [\mathcal{D}(A^*)]', \qquad y(0) = y_0 \in Y, \tag{1.2.1}$$

where $U$ and $Y$ are, respectively, control and state Hilbert spaces, and where:

(i) the operator $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup $e^{At}$ on $Y$, $t \geqslant 0$;

(ii) $B$ is an "unbounded" operator $U \to Y$ satisfying $B \in \mathcal{L}(U; [\mathcal{D}(A^*)]')$ or equivalently, $A^{-1}B \in \mathcal{L}(U; Y)$. Above, as well as in (1.2.1), $[\mathcal{D}(A^*)]'$ denotes the dual space with respect to the pivot space $Y$, of the domain $\mathcal{D}(A^*)$ of the $Y$-adjoint $A^*$ of $A$. Without loss of generality, we take $A^{-1} \in \mathcal{L}(Y)$.

Many examples of these abstract models are given under one cover in [5, 39] and [46, Vols. 1-2]. They include the case of first order hyperbolic systems quoted before, where again the need for an abstract model came from boundary PDE control theory, and was not available in the purely PDE theory *per se.* See [46, Subsect. 10.6] and [48, Subsect. 4.1]. Accordingly, having accomplished a first abstract unification of many dynamical PDE mixed problems, it was natural to attempt to extract—wherever possible—additional, more in-depth, common 'abstract properties,' shared by sufficiently many classes of PDE mixed problems. For the purpose of this note, we focus on three "abstract properties:" (optimal) regularity, exact controllability, and uniform stabilization.

**Regularity.**

The variation of the parameter formula for (1.2.1) is

$$y(t) = e^{At}y_0 + (Lu)(t); \tag{1.2.2a}$$

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau;$$
$$L_T u = (Lu)(T) = \int_0^T e^{A(T-t)} Bu(t) dt. \tag{1.2.2b}$$

*Per se*, the abstract differential equation (1.2.1) is not the critical object of investigation. It is good to have it, inasmuch as it yields (1.2.2). The key element that defines the crucial feature of a particular PDE mixed problem, is, however, the regularity of the operators $L$ and $L_T$. This is what was referred to above as "interior regularity:" the control $u$ acts on the boundary, while $Lu$ is the corresponding solution acting in the interior. Accordingly, this pursued line of investigation brought about a second, abstract realization [26, 27, 28, 45]: that of determining the "best" function space $Y$ for each class of mixed hyperbolic and Petrowski type problems, such that the following interior regularity property holds:

$$L : \text{ continuous } L_2(0, T; U) \to C([0, T]; Y), \tag{1.2.3}$$

for one, hence for all positive, finite $T$. Presently, such space $Y$ is explicitly identified in most (but by no means all) of the mixed PDE problems of hyperbolic or Petrowski type. [The case $Y = [\mathcal{D}(A^*)]'$ is always true in the present setting, and not much informative, save for offering a back-up result for (1.2.1).] An equivalent (dual) formulation is given in (1.2.4) below [27, 28, 8].

For the mixed PDE classes under considerations, achieving the regularity property (1.2.3) with the "best" function space $Y$ is the accomplishment of hard analysis PDE energy methods, tuned to the specific combination of PDE and boundary control, which first produce, for each such individual combination, a PDE-estimate for the corresponding dual PDE problem. The precursor was the multidimensional wave equation with Dirichlet control [26, 27, 22]. All such *a priori* estimates thus obtained on an individual basis admit the following "abstract version:"

$$L_T^* \equiv B^* e^{A^* t} : \text{ continuous } Y \to L_2(0, T; U), \tag{1.2.4}$$

where $L_T$ is defined by (1.2.2b) [26, 27, 22].

In PDE mixed problems, property (1.2.4) is a (sharp) 'trace regularity property' of the boundary homogeneous problem, which is dual to the corresponding map $L_T$ in (1.2.2b): from the $L_2(0, T; U)$-boundary control to the PDE solution at time $T$, see many examples in the books [39, 46]. Indeed, such a PDE estimate is both nontrivial and unexpected, and typically yields a finite *gain* (often $\frac{1}{2}$) *in the space regularity* of the solution trace over a *formal* application of trace theory to the optimal interior regularity of the

PDE solution. Some PDE circles have come to call it "hidden regularity," and with good reasons. It was first discovered in the case of the wave equation with Dirichlet control [27]. It should be referred to, more precisely, as "hidden sharp regularity."

Only after the fact, if one so wishes, functional analytic methods can be brought into the analysis to show that, in fact, the abstract trace regularity (1.2.4) is equivalent to the interior regularity property (1.2.3) [27, 28, 8]. [Needless to say, this can actually be done also on a case-by-case basis for each PDE class.] This is the spirit of abstract, unifying treatments of optimal control problems for PDE subject to boundary (and point) control, that can be found in books such as [39, 5] and [46, Vol. 2]. As mentioned above, the regularity (1.1.4) is *equivalent* to the regularity (1.2.3) by a duality argument [27, 28, 8].

**Surjectivity of $L_T$, or exact controllability.**

In a similar vein, we can describe the second abstract dynamic property of model (1.2.1) or (1.2.2); namely, the property that the input-solution operator $L_T$, defined in (1.2.2b), satisfies

$$L_T \text{ be surjective}: \; L_2(0,T;U) \to \text{ onto } Y_1, \tag{1.2.5}$$

where $Y_1 \subset Y$. In the most desirable case $Y_1$ is the same space $Y$ as in (1.2.3). This is, in fact, often the case with hyperbolic and Petrowski type systems, but is by no means always true [example, second order hyperbolic equations with Neumann control, Euler–Bernoulli plate equations with control in "high" boundary conditions]. For time reversible dynamics such as the hyperbolic and Petrowski type systems under consideration, the functional analytic property (1.2.5) is re-labelled "exact controllability in $Y_1$ at $t = T$" in the PDE control theory literature. By a standard functional analysis result [77, p. 237], property (1.2.5) is equivalent by duality to the following so-called "abstract continuous observability" estimate:

$$\|L_T^* z\| \geqslant C_T \|z\| \text{ or } \int_0^T \|B^* e^{A^* t} x\|_U^2 dt \geqslant C_T \|x\|_{Y_1}^2 \quad \forall\, x \in Y_1, \tag{1.2.6}$$

perhaps only for $T$ sufficiently large in hyperbolic problems with finite speed of propagation, which we recognize as being the reverse inequality of (1.2.4), at least when $Y_1 = Y$, and $T$ is large.

The crux of the matter begins now: How does one establish the validity of characterization (1.2.6) for exact controllability in the appropriate function spaces $U$ and $Y_1$—in particular, if we can take $Y_1 = Y$—for the classes of multidimensional hyperbolic and Petrowski type PDE with boundary control? The answer is: by appropriate PDE-energy methods, tuned to each special class/problem. Negative results on exact controllability are given in [79, 85].

**Uniform stabilization.**

One may repeat the same set of considerations, in the same spirit, when it comes to establishing uniform stabilization of an originally conservative hyperbolic or Petrowski type system, by means of a suitable boundary dissipation. The abstract characterization is an inverse type inequality such as (1.2.6), except that it refers now to the boundary *dissipative* mixed PDE problem, not the boundary *homogeneous conservative PDE* problem. The particular abstract inequality will be given below in (1.3.12), in the context under discussion. Typically, establishing the uniform stabilization inequality for *the class* of hyperbolic, or Petrowski type PDEs *under discussion* is more challenging, sometimes by much, than obtaining the corresponding specialization of the continuous observability inequality (1.2.6). Negative results on uniform stabilization are given in [82, 84].

**Constant coefficients versus variable coefficients.**

Here the situation regarding the aforementioned three properties is clear:

(1) *Regularity* of solutions (PDE specialization of the "abstract trace regularity" (1.2.4)): variable coefficients (in time and space) in the principal part and in the lower order terms are *benign* (provided they are suitably smooth). That is, the PDE tricks (energy methods) which work in the case of constant coefficients in the principal part and no lower order terms, when applied to the more general variable coefficient case, produce lower order terms that can be readily absorbed in the sought-after estimates. This has been known since the 1986 paper [22] on second order hyperbolic equations.

(2) *Continuous observability and uniform stabilization estimates* (PDE specialization of (1.2.6) and (1.3.12)). Here, the situation is drastically different. Even the presence of energy-level terms, particularly with variable coefficients, represents a major additional difficulty. A further serious level of difficulty is encountered in the case of variable coefficients (in space) in the principal part. To overcome these serious challenges, a few new methods have been introduced. The authors have favored energy methods in a suitable Riemannian metric defined in terms of the coefficients $a_{ij}(x)$ of the principal part $\mathbb{A}$ in (1.1.0). For Riemannian geometric methods in exact controllability and uniform stabilization, we refer to [10], [53]–[55], [89, 91, 92], [95]–[97].

## *1.3 Abstract setting encompassing the second order and first order (in time) hyperbolic and Petrowski type PDEs of the present paper*

The correct abstract setting for the present classes of PDE-problems here considered was given in [19], [46, Chapt. 7, p. 663], and used in [48].

**A second order equation setting.**

Let $H$, $U$ be Hilbert spaces, and let

(h.1) $\mathcal{A} : H \supset \mathcal{D}(\mathcal{A}) \to H$ be a positive self-adjoint operator;

(h.2) $\mathcal{B} \in \mathcal{L}(U; [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$; equivalently, $\mathcal{A}^{-\frac{1}{2}}\mathcal{B} \in \mathcal{L}(U; H)$.

We consider the open-loop control system

$$v_{tt} + \mathcal{A}v = \mathcal{B}u, \quad v(0) = v_0, \; v_t(0) = v_1, \tag{1.3.1}$$

as well as the corresponding closed-loop, dissipative feedback system

$$w_{tt} + \mathcal{A}w + \mathcal{B}\mathcal{B}^* w_t = 0, \quad w(0) = w_0, \; w_t(0) = w_1. \tag{1.3.2}$$

We rewrite (1.3.1) and (1.3.2) as first order systems of the form (1.2.1) in the space $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times H$:

$$\frac{d}{dt}\begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} = A \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} + Bu; \quad \frac{d}{dt}\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = A_F \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix}; \tag{1.3.3}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \; A_F = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B}\mathcal{B}^* \end{bmatrix} = A - BB^*, \; B = \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}, \tag{1.3.4}$$

with obvious domains. The operator $A_F$ is maximal dissipative and thus the generator of a s.c. contraction semigroup $e^{A_F t}$, $t \geqslant 0$, on $Y$ [46, Proposition 7.6.2.1, p. 664].

Setting $y(t) = [w(t), w_t(t)]$, $y_0 = [w_0, w_1]$, we have that the variation of parameter system for the $w$-problem is

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = y(t) = e^{A_F t}y_0 = e^{At}y_0 - \int_0^t e^{A(t-\tau)}BB^* e^{A_F \tau} y_0 d\tau \tag{1.3.5a}$$

$$= e^{At}y_0 - \{L(B^* e^{A_F \cdot} y_0)\}(t), \tag{1.3.5b}$$

recalling the operator $L$ defined in (1.2.2b).

### A first order equation setting

We now consider a first order model with skew-adjoint generator. Let $Y$ and $U$ be two Hilbert spaces. The basic setting is now as follows:

(a.1) $A = -A^*$ is a skew-adjoint operator $Y \supset \mathcal{D}(A) \to Y$, so that $A = iS$, where $S$ is a self-adjoint operator on $Y$, which (essentially without loss of generality) we take positive definite (as in the case of the Schrödinger equation of Sect. 6 below).

Accordingly, the fractional powers of $S$, $A$, $A^*$ are well defined.

(a.2) $B$ is a linear operator $U \to [\mathcal{D}(A^{*\frac{1}{2}})]'$, duality with respect to $Y$ as a pivot space; equivalently, $Q \equiv A^{-\frac{1}{2}}B \in \mathcal{L}(U;Y)$ and $B^*A^{*-\frac{1}{2}} \in \mathcal{L}(Y;U)$.

The first order setting under (a.1) and (a.2), includes the second order setting under (h.1) and (h.2).

Under assumptions (a.1) and (a.2), we consider the operator $A_F : Y \supset \mathcal{D}(A_F) \to Y$ defined by

$$A_F x = [A - BB^*]x;\ x \in \mathcal{D}(A_F) = \{x \in Y : [A - BB^*]x \in Y\}. \tag{1.3.6}$$

**Proposition 1.3.1.** *Under assumptions* (a.1) *and* (a.2) *above, we have, with reference to* (1.3.6):

(i) *We have*

$$\mathcal{D}(A_F) = A^{-\frac{1}{2}}[I - iQQ^*]^{-1}A^{-\frac{1}{2}}Y \subset \mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B^*); \tag{1.3.7a}$$

$$A_F^{-1} = A^{-\frac{1}{2}}[I - iQQ^*]^{-1}A^{-\frac{1}{2}} \in \mathcal{L}(Y). \tag{1.3.7b}$$

(ii) *The operator $A_F$ is dissipative*; *in fact, maximal dissipative, and hence the generator of a s.c. contraction semigroup $e^{A_F t}$ on $Y$, $t \geqslant 0$.* [*Similarly, the $Y$-adjoint $A_F^*$ is the generator of a s.c. contraction semigroup on $Y$, with $A_F^{*-1}$ given by the same expression* (2.7b) *with* "+" *sign rather than* "−" *sign for the operator in the middle.*]

(iii) *Hence the abstract first order, closed-loop equation*

$$\dot{y} = (A - BB^*)y, \qquad y(0) = y_0 \in Y \tag{1.3.8a}$$

(*obtained from the open-loop equation*

$$\dot{\eta} = A\eta + Bu \tag{1.3.8b}$$

*with feedback* $u = -B^*y$) *admits the unique solution* $y(t) = e^{A_F t}y_0$, $t \geqslant 0$, *satisfying the energy identity*

$$\|y(t)\|_Y^2 + 2\int_s^t \|B^*y(\tau)\|_U^2 d\tau = \|y_0(s)\|_U^2, \quad 0‘s \leqslant t. \tag{1.3.8c}$$

*In particular,*

$$\int_0^\infty \|B^*y(\tau)\|_U^2 d\tau = \frac{1}{2}\|y_0\|_Y^2. \tag{1.3.8d}$$

*Proof.* (i) Let $x \in \mathcal{D}(A_F)$. Then we can write

$$\begin{aligned} A_F x = [A - BB^*]x &= A^{\frac{1}{2}}[I - (A^{-\frac{1}{2}}B)(B^*A^{-\frac{1}{2}})]A^{\frac{1}{2}}x \\ &= A^{\frac{1}{2}}[I - iQQ^*]A^{-\frac{1}{2}}x = f \in Y, \end{aligned} \tag{1.3.9}$$

with $Q \equiv A^{-\frac{1}{2}}B \in \mathcal{L}(U;Y)$ by assumption, and $A^* \equiv B^*A^{*-\frac{1}{2}} \in \mathcal{L}(Y;U)$, its dual or conjugate. Here, we have used (a.1): $A^* = -A$, so that $A^{*\frac{1}{2}} = iA^{\frac{1}{2}}$, hence $A^{-\frac{1}{2}} = iA^{-\frac{1}{2}}$, finally $B^*A^{-\frac{1}{2}} = iB^*A^{*-\frac{1}{2}} = iQ^*$. It is clear that the operator $[I - iQQ^*]$, where $QQ^* \in \mathcal{L}(Y)$ is nonnegative, self-adjoint on $Y$, is boundedly invertible on $Y$. Thus, (1.3.9) yields

$$x = A_F^{-1}f = A^{-\frac{1}{2}}[I - iQQ^*]^{-1}A^{-\frac{1}{2}}f \in \mathcal{D}(A_F), \ f \in Y, \tag{1.3.10}$$

and (1.3.7a–b) is proved. Then the identity in (1.3.7a) plainly shows that $\mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}})$, while $\mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B^*)$ by assumption (a.2). Part (i) is proved.

(ii) We next show that $A_F$ is dissipative. Let $x \in \mathcal{D}(A_F)$. Thus, $x \in \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{*\frac{1}{2}}) \subset \mathcal{D}(B^*)$ by part (i). Hence we can write if ( , ) is the $Y$-inner product:

$$\operatorname{Re}(A_F x, x) = \operatorname{Re}([A - BB^*]x, x) = \operatorname{Re}(x, x) - \|B^*x\|^2 \tag{1.3.11a}$$

$$\leqslant -\|B^*x\|^2 \leqslant 0 \quad \forall\, x \in \mathcal{D}(A_F), \tag{1.3.11b}$$

since $\operatorname{Re}(Ax, x) = \operatorname{Re}\{-i\|A^{\frac{1}{2}}x\|^2\} = 0$, where each term in (1.2.11a-b) is well-defined. Thus, $A_F$ is dissipative.

Finally, since $A_F^{-1} \in \mathcal{L}(Y)$ by part (i), then $(\lambda_0 - A_F)^{-1} \in \mathcal{L}(Y)$ as well for a suitable small $\lambda_0 > 0$, and then the range condition: range$(\lambda_0 - A_F) = Y$ is satisfied, so that $A_F$ is maximal dissipative. By the Lumer–Phillips theorem [69, p. 14], $A_F$ is the generator of a s.c. contraction semigroup on $Y$. The same argument shows that $A_F^*$ is maximal dissipative.

(iii) A standard energy method: one takes the $Y$-inner product of (1.3.8a) with $y$, uses $\frac{1}{2}\frac{d}{dt}\|y(t)\|_Y^2 = \text{Re}(y_t, y)_Y$, as well as $\text{Re}(Ay, y) = 0$ for $A$ skew-adjoint, and integrates in time. □

*Remark 1.3.1.* One can, of course, extend the range of Proposition 1.3.1, by adding to $A$ a suitable perturbation $P$: either $P \in \mathcal{L}(Y)$ or else $P$ relatively bounded dissipative perturbations as in known results [69, Corollary 3.3, Theorem 3.4, p. 82–83] for instance, and still obtain that $[(A+P) - BB^*]$ is the generator of a s.c. semigroup (of contractions in the last two cases). □

**The main thrust of references [48, 49].**

The main thrust of authors' prior efforts in [48, 49] (two references very difficult to obtain, due to problems of the journal, which was also discontinued for some time) dealt with a question which was raised in [9, Theorem 3] only in connection with the second order system (1.3.1), (1.3.2) subject to assumptions (h.1) and (h.2) that precede (1.3.1). However, in view of [48, Proposition 1.3.1] likewise extended the same question to the first order systems (1.3.8a-b) subject to assumptions (a.1) and (a.2) that precede Proposition 1.3.1. For both problems we have $A^* = -A$, the skew-adjoint property of the free dynamics generator. With reference to the systems (1.3.1), (1.3.2), the question was: Is it true that exact controllability of (1.3.1) on the state space $Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times H$ by means of $L_2(0,T;U)$-controls is equivalent to uniform stabilization of (1.3.2) on the same space $Y$? Reference [48] extended this question also in reference to the systems (1.3.8a-b), in order to include, for instance, also the Schrödinger equation case of Sect. 4. Henceforth, $\{A, B, A_F, Y, U\}$ refer either to (1.3.5) or to (1.3.8), indifferently. Quantitatively, we may reformulate the above equation as follows: Is the continuous observability inequality (1.2.6) [which characterizes exact controllability of (1.2.1) with $A$ and $B$ as in (1.3.4) or as in (1.3.8b)] equivalent to the inequality

$$\int_0^T \|B^* e^{A_F t} x\|_U^2 dt \geqslant c_T \|e^{A_F T} x\|_Y^2 \qquad \forall\, x \in Y, \tag{1.3.12}$$

which characterizes the uniform stability of the $w$-problem (1.3.2) or the $y$-problem (1.3.8a)? In our case, $A$ is skew-adjoint $A^* = -A$. Thus, exact controllability of $\{A, B\}$ (that is of (1.3.1) or (1.3.8a)) over $[0,T]$ is equivalent to exact controllability of $\{A^*, B\}$ over $[0,T]$. In other words, in our case, the inequality (1.2.6) is equivalent to

$$\int_0^T \|B^* e^{At} x\|_U^2 dt \geqslant c_T \|x\|_Y^2 \qquad \forall\, x \in Y. \tag{1.3.13}$$

Thus, the present question is rephrased now as follows: Is the inequality (1.3.12) equivalent to the inequality (1.3.13)?

In one direction, the implication: uniform stabilization of (1.3.1) or (1.3.8b) [i.e., (1.3.12)] $\rightarrow$ exact controllability of (1.3.1) or (1.3.8b) [i.e., (1.3.13)] was shown by Russell [72, 73] some 30 years ago, by virtue of a clean soft argument.

In the opposite direction, we have the following:

**Claim 1.3.1** ([48]). *With reference to the second order equations* (1.3.1), (1.3.2) [*respectively, the first order equations* (1.3.8a-b)*, assume the preceding assumptions* (h.1) *and* (h.2) [*respectively,* (a.1) *and* (a.2)]. *Then, the implication*: *the exact controllability of* (1.3.1) *or* (1.3.8b) [*i.e.,* (1.3.13)] $\Rightarrow$ *uniform stabilization of* (1.3.2) *or of* (1.3.8a) [*i.e.,* (1.3.12)] *holds, if one adds the assumption that*

$$\text{the operator } B^*L\text{: continuous } L_2(0,T;U) \rightarrow L_2(0,T;U). \tag{1.3.14}$$

*Remark 1.3.2.* We remark that if $B$ is, in particular, a bounded operator, $B \in \mathcal{L}(U;Y)$, then [condition (1.2.3) and] condition (1.3.14) is, *a fortiori*, satisfied. Thus, in this case, exact controllability of (1.3.1) or (1.3.8b) implies (and is implied by [72, 73]) uniform stabilization. We recover (with the simple proof of Sect. 3) a 30-year-old well-known result of [74] (based on the same finite-dimensional proof of [66]). □

Reference [48] gave an extension of Claim 1.3.1, which involved a *nonlinear* feedback version. This is reported below.

### A first nonlinear extension of Claim 1.3.1.

In place of Equation (1.3.8a) (hence (1.3.2)), we consider the following nonlinear version

$$y_t = Ay - Bf(B^*y), \quad y(0) = y_0 \in Y, \tag{1.3.15a}$$

under the same assumptions (a.1) for $A$ and (a.2) for $B$, where $f$ is a monotone increasing, continuous function on $U$. It is known [19, 21] that $A - Bf(B^*)$ *generates a nonlinear semigroup of contractions*—say $S_F(t)$—which yield the following variation of parameter formula for (1.3.15):

$$y_t = S_f(t)y_0 = e^{At}y_0 - \{L(f(B^*S_F(\,\cdot\,)y_0)\}(t), \tag{1.3.15b}$$

and obeys the energy identity

$$\|S_F(T)y_0\|_Y^2 = \|y(T)\|_Y^2 = \|y(0)\|_Y^2 - 2\int_0^T (f(B^*y), B^*y)_U dt. \tag{1.3.16}$$

**Proposition 1.3.2.** *In addition to the standing assumptions, we assume that*

(i) *The operator* $B^*L$ *is continuous* $L_2(0,T;U) \to L_2(0,T;U)$, *as in* (1.3.14);

(ii) $m\|u\|_U^2 \leqslant (f(u),u)_U$; $\|f(u)\|_U \leqslant M\|u\|_U$ *for all* $u \in U$.

*Then the exact controllability of* $(A,B)$ *implies the exponential stability of* $S_F(t)$, *i.e., there exist positive constants* $C, \omega > 0$ *such that the solution of* (1.3.14) *satisfies*

$$\|y(t)\|_Y^2 \leqslant Ce^{-\omega t}\|y_0\|_Y^2. \tag{1.3.17}$$

*Proof. Step* 1. We first show that for any $y_0 \in Y$, we have via assumption (i) = (1.3.14) and (ii),

$$\|B^* e^{A\cdot} y_0\|_{L_2(0,T;U)} \leqslant (1 + k_T M)\|B^* S_F(\,\cdot\,)y_0\|_{L_2(0,T;U)}, \tag{1.3.18}$$

where $k_T = |||B^*L|||$ in the uniform operator norm of $\mathcal{L}(L_2(0,T;U))$. Indeed, (1.3.18) stems readily from (1.3.15), which yields

$$B^* e^{At} y_0 = B^* S_F(t)y_0 + \{[B^*L]f(B^* S_F(\,\cdot\,)y_0\}(t). \tag{1.3.19}$$

Hence, invoking assumption (1.3.14) on $B^*L$, we see that (1.3.19) along with the upper bound on $f$ in the RHS of (ii) at once implies (1.3.18).

*Step* 2. The exact controllability assumption on the pair $\{A, B\}$, equivalently on the pair $\{A^*, B\}$, guarantees characterization (1.3.13). This, combined with (1.3.18), yields then, for any $y_0 \in Y$:

$$\|S_F(T)y_0\|_Y \leqslant \|y_0\|_Y^2 \leqslant C_T \int_0^T \|B^* e^{At} y_0\|_U^2 dt$$

$$\leqslant C_T(1 + k_T M) \int_0^T \|B^* S_F(t)y_0\|_U^2 dt, \tag{1.3.20}$$

where the first inequality is due to (1.3.16). [In the linear case, $f(u) = u$, the proof stops here, see (1.3.12).]

*Step* 3. The energy identity (1.3.16), when combined with (1.3.20) and (i), gives

$$\|S_F(T)y_0\|_Y^2$$
$$\leqslant C_T(1 + k_T M) \int_0^T \|B^* S_F(t)y_0\|_U^2 dt + 2\int_0^T (B^* S_F(t)y_0, f(B^* S_F(t)y_0))_U dt$$

$$\leqslant \left(C_T(1+k_T M)m^{-1}+2\right)\int_0^T (B^* S_F(t)y_0, f(B^* S_F(t)y_0))_U \, dt$$

$$= \left(C_T(1+k_T M)m^{-1}+2\right)\left(\|S_F(0)y_0\|_Y^2 - \|S_F(t)y_0\|_Y^2\right). \qquad (1.3.21)$$

The above identity implies that $\|S_F(T)\|_Y \leqslant \gamma < 1$, which, in turn, implies exponential decays for the semigroup. The proof of Proposition 1.3.2 is complete. □

References [48, 49] argued convincingly, on the base of a wealth of explicit hyperbolic or Petrowski PDE illustrations, that *showing uniform boundary stabilization* for such classes can be done *more conveniently directly*, by establishing the concrete version (on a case-by-case basis) of the abstract inequality (1.3.12) [which characterizes the uniform stabilization of the $w$-problem (1.3.2), or the $y$-problem (1.3.8a)], rather *than seeking to first establish the boundedness* (1.3.14) of the operator $B^*L$ and then invoke (even in the linear case $f(u) = u =$ identity) Proposition 1.3.2. This is so since showing (1.3.14) for $B^*L$ is either a more challenging task [as in the case of the wave equation with Dirichlet boundary control of the subsequent Sect. 4]; or else the proof of a result such as (1.3.14), or technically comparable to it and very close to it, is *actually built in* to existing proofs of regularity/exact controllability/uniform stabilization of *some* (surely, not all) Petrowski type systems, for example, the case of the Schrödinger equation with Dirichlet control of Sect. 6 below].

More critically, there are hyperbolic/Petrowski type PDE problems (as those identified in [48, 49], and at the end of Sect. 2), where the boundedness condition (1.3.14) on $B^*L$ *fails*, yet the corresponding uniform stabilization results hold, and had been known since the early '80s. In short: assumption (1.3.14) is far from being a necessary condition for uniform stabilization within the hyperbolic/Petrowski classes of PDEs. Negative examples include:

(i) the wave equation with feedback dissipation in the Neumann boundary condition in the finite energy space $H^1(\Omega) \times L_2(\Omega)$ [48, Sect. 6];

(ii) the Schrödinger equation with feedback dissipation also in the Neumann boundary condition in the state space $H^1(\Omega)$ [48, Sect. 8], and the present Sects. 9.1 and 9.2.

As explained in the introductory Subsect. 1.1, *our main aim in the present paper is different*: Having established [48, 49] that a PDE problem possesses the property that the boundary control → boundary observation map $B^*L$ is bounded as asserted in (1.3.14), what is the physical/control-theoretic meaning; what are the positive implications that can be derived? The next sections answer these questions on a case-by-case basis.

# 2 Open Loop Problem (1.2.1): From $B^*L$ Bounded to $L$ Bounded, Equivalently $B^*e^{A^*\cdot}$ Bounded

We return to the abstract equation (1.2.1)

$$\dot{y} = Ay + Bu, \text{ in } [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y, \tag{2.1}$$

where $U$ and $Y$ are the control and the state (Hilbert) spaces, in the setting of Subsect. 1.3 which encompasses second order models (for hyperbolic and Petrowski type PDEs) under the abstract assumptions (h.1) and (h.2) specified there, as well as first order models (such as the Schrödinger equation and first order hyperbolic systems) under the abstract assumptions (a.1) and (a.2). The star $*$ in the adjoint $B^*$ refers to the assigned spaces $U$ and $Y$. Thus, the operator $B^*L$, *per se*, refers unequivocally to the space $U$ and $Y$, which have helped its definition. The present section is devoted to the following set of implications for the open-loop problem. The operators $L$ and $L_T$ are defined in (1.2.2b), and provide the solution to (2.1) according to formula (1.2.2a).

**Theorem 2.1 ([49]).** *We consider the problem* (2.1) *either under assumptions* (h.1) *and* (h.2)*, or else* (a.1)*,* (a.2) *of Subsect.* 1.3*, with respect to the control space $U$ and the state space $Y$. Explicitly, this means that $A = -A^*$ is skew-adjoint operator $Y \supset \mathcal{D}(A) \to Y$, thus generator of a unitary group on $Y$ with $e^{A^*t} = e^{-At}, \quad t \in \mathbb{R}$, and $A^{-\frac{1}{2}}B \in \mathcal{L}(U; Y)$.*

*Moreover, assume that*

$$B^*L \in \mathcal{L}(L_2(0,T;U)). \tag{2.2}$$

*Then, in fact,*

$$L \text{ is continuous: } L_2(0,T;U) \to C([0,T];Y), \tag{2.3}$$

*where* (2.3) *is equivalent to*

$$B^*e^{A^*t} = B^*e^{-At} : \text{ continuous } Y \to L_2(0,T;U). \tag{2.4}$$

*Proof.* We report two proofs [49].

*Proof #1.* Start with $u$ smooth, say $u \in C^1([0,T];U), u(0) = 0$, so that by parts

$$x(t) = \int_0^t e^{-As}Bu(s)ds \in C^1([0,T];\mathcal{D}(A^{\frac{1}{2}})). \tag{2.5}$$

By skew-adjointness of $A = -A^*$, as well as by (2.2), we then estimate from (2.5), recalling (1.1.2):

$$C_T ||u||^2_{L_2(0,T;U)}$$

$$\geqslant \int_0^T ((B^*Lu)(t), u(t))_U \, dt = \int_0^T \left( \int_0^t e^{A(t-s)} Bu(s) ds, Bu(t) \right)_Y dt \tag{2.6}$$

$$= \int_0^T \left( \int_0^t e^{-As} Bu(s) ds, e^{-At} Bu(t) \right)_Y dt = \int_0^T \left( x(t), \frac{d}{dt} x(t) \right)_Y dt \tag{2.7}$$

$$= \frac{1}{2} \int_0^T \frac{d}{dt} (x(t), x(t))_Y \, dt = \frac{1}{2} \|x(T)\|_Y^2 = \frac{1}{2} \left\| \int_0^T e^{-As} Bu(s) ds \right\|_Y^2 \tag{2.8}$$

$$= \frac{1}{2} \left\| e^{-AT} \int_0^T e^{A(T-s)} Bu(s) ds \right\|_Y^2 \sim \left\| \int_0^T e^{A(T-s)} Bu(s) ds \right\|_Y^2 \tag{2.9}$$

$$= c_T \|L_T u\|_Y^2 . \tag{2.10}$$

Then the estimate in (2.10) can be extended to all $u \in L_2(0,T;U)$,

$$\|L_T u\|_Y^2 \leqslant \text{const}_T \|u\|^2_{L_2(0,T;U)} \quad \forall \, u \in L_2(0,T;U). \tag{2.11}$$

Then it is well known [8, 30], [46, p. 648] that (2.11) yields (2.3). □

*Remark 2.3.* Theorem 2.1 can be extended to $A$ of the form $A = iS + kI$, with $S$ a self-adjoint operator on $Y$ and $k \in \mathbb{R}$, so that $A^* = -A + 2kI$, and $e^{A^*t} = e^{-At} e^{2kt}$. In this case, we start with $(B^*Lu, u_1)_U$, with $u_1 = e^{-2kt} u(t)$, $u \in L_2(0,T;U)$. □

*Proof #2.* An alternative, perhaps more insightful, proof of Theorem 2.1 is as follows. For $L$ in (1.2.2b) and its adjoint $L^*$ we have [46] for $u$ smooth

$$(B^*Lu)(t) = \int_0^t B^* e^{A(t-\tau)} Bu(\tau) d\tau; \tag{2.12}$$

$$(L^*Bu)(t) = \int_t^T B^* e^{A^*(\tau-t)} Bu(\tau) d\tau = \int_t^T B^* e^{A(t-\tau)} Bu(\tau) d\tau, \tag{2.13}$$

using the skew-adjoint assumption $A^* = -A$. Thus, adding up (2.12) and (2.13) yields, using again skew-adjointness:

$$(B^*Lu)(t) + (L^*Bu)(t) = \int_0^T B^* e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.14}$$

$$= B^* e^{A(t-T)} \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau \tag{2.15}$$

$$= B^* e^{A(T-t)} \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau. \tag{2.16}$$

Finally, recalling $L_T$ from (1.2.2b) and its adjoint [46], we rewrite (2.16) in the following form:

$$B^*Lu + L^*Bu = L_T^* L_T u, \quad u \in L_2(0,T;U). \tag{2.17}$$

[We note that by taking the $L_2(0,T;U)$-inner product of (2.17) with $u$, we obtain

$$2(Lu, Bu)_{L_2(0,T;U)} = \|L_T u\|_Y^2, \quad u \in L_2(0,T;U), \tag{2.18}$$

thus recovering the identity buried in (2.9).] Equation (2.18) shows, again, the implication (2.2) ⇒ (2.11), hence (2.3), as is well known [30], [46, p. 648]. The proof #2 is complete. □

Finally, the equivalence between (2.3) and (2.4) is well known [8, 30, 46] (by duality). The proof of Theorem 2.1 is complete. □

**Corollary 2.1.** *Assume that the assumptions of Theorem* 2.1 *regarding the open-loop problem* (2.1) (= (1.2.1)) *hold. Then the following regularity result holds*: *The map*

$$\{y_0, u\} \in Y \times L_2(0,T;U) \\ \to B^*y = B^*Lu + B^*e^{At}y_0 \in L_2(0,T;U) \tag{2.19}$$

*is continuous.*

**Consequences.**

*PDE mixed problems where the regularity* (2.3) *of $L$ is false*: *a fortiori, the regularity* (2.2) *of $B^*L$ is false* (*yet uniform stabilization of the corresponding dissipative problem holds*). *Alternative strategy for uniform stabilization of the corresponding nonlinear boundary dissipative problems.*

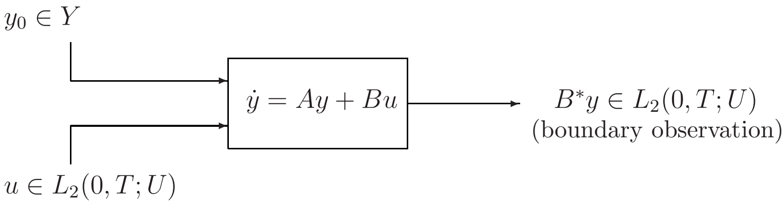


**Fig. 1** Open-loop regularity {I.C., boundary control} $\to$ boundary observation, under the assumption $B^*L \in \mathcal{L}(L_2(0,T;U)$, for the system (1.2.1), in the setting (a.1), (a.2) of Subsect. 1.3

These PDE-mixed problems were already noted in [48, Sect. 6 (Examples 6.2 and 6.3) and Sect. 8] and [49, Remark 4.5]. These are all PDE-mixed problems where $B^*L$ fails *a fortiori* to satisfy the regularity property (2.2), *yet* uniform stabilization (under appropriate geometrical conditions) does hold in each case, as has been known since the early 80's. This re-confirms one of the points of [48] that assumption (2.3) for $B^*L$ made in Theorem 3.1 to yield uniform stabilization (generalizing Claim 1.3.1 of Subsect. 1.3) is generally too strong (and no advantage to check, when it holds); even in the linear case. In these (and other PDE) cases, there is, however, an *alternative strategy* to obtain uniform stabilization with nonlinear boundary feedback: namely, along the general approach of [24], originally carried out for waves, which was already exported to other dynamics: shells [47], Schrödinger equations on the state space $L_2(\Omega)$ with Neumann feedback dissipative control [51]. (See subsequent Sect. 9.) We briefly indicate the list of the aforementioned PDE problems.

**Example #1.** The open-loop wave equation in $\Omega$, dim $\Omega \geqslant 2$, with Neumann boundary control $g \in L_2(0,T;L_2(\Gamma_1)) \equiv L_2(\Sigma_1)$, and its corresponding closed-loop dissipative system:

$$\begin{cases} v_{tt} = \Delta v \\ v(0,\cdot) = v_0,\ v_t(0,\cdot) = v_1 \\ v|_{\Sigma_0} = 0 \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma_1} = g \end{cases} \quad \begin{cases} w_{tt} = \Delta w & \text{in } Q; & (2.20\text{a}) \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (2.20\text{b}) \\ w|_{\Sigma_0} = 0 & & (2.20\text{c}) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = w_t & \text{in } \Sigma, & (2.20\text{d}) \end{cases}$$

with $Q = (0,T] \times \Omega$, $\Sigma_i = (0,T] \times \Gamma_i$, $i = 0,1$; $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \neq \phi$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \phi$; $h \cdot \nu \leqslant$ on $\Gamma_0$ for a coercive smooth vector field $h$ on $\Omega$. For the theory of sharp/optimal regularity of the mixed $v$-problem we refer to [34, 36, 38, 45], [46, Subsect. 9.4, p. 857 for dim $\Omega = 1$], [76].

In particular, for dim $\Omega \geqslant 2$, $g \in L_2(0, T; L_2(\Gamma))$ does $NOT$ imply $v \in C[0, T]; H^1(\Omega))$, not even $v \in H^{\frac{3}{4}+\varepsilon}(Q)$, for all $\varepsilon > 0$ [and $\varepsilon = 0$ only for flat boundary (parallelepiped)]. In fact, $\{v, v_t\} \in C([0, T]; H^{\frac{2}{3}}(\Omega) \times H^{-\frac{1}{3}}(\Omega))$ is the best result, for general $\Omega$. See counterexample in [36, p. 294]. Uniform stabilization of the $w$-problem is given in [83]. Exact controllability of the corresponding nonlinear problem is given in [88]. Exact controllability for the corresponding boundary value problem is given in [88].

**Example #2.** The Euler–Bernoulli plate model in dim $\Omega = 2$, with free boundary condition:

$$\begin{cases} v_{tt} + \Delta^2 v + v = 0 & \text{in } (0, T] \times \Omega \equiv Q; \quad (2.21a) \\ v(0, \cdot) = v_0, \ v_t(0, \cdot) = v_1 & \text{in } \Omega; \quad (2.21b) \\ [\Delta v + (1 - \eta) B_1 v]_\Sigma = 0 & \text{in } (0, T] \times \Gamma \equiv \Sigma; \quad (2.21c) \\ \left[\dfrac{\partial \Delta v}{\partial \nu} + (1 - \eta) B_2 v\right]_\Sigma = g \text{ in } \Sigma, & \quad (2.21d) \end{cases}$$

where $0 < \eta < 1$ is the Poisson modulus and $B_1$ and $B_2$ are the usual boundary operators, defined, say, in [16, 17], [46, Vol. 1, p. 249]. Here, with reference to the problem (2.21a–d), the space of finite energy is $Y \equiv H^2(\Omega) \times L_2(\Omega)$. Yet, for dim $\Omega \geqslant 2$ the map $g \rightarrow Lg = \{v, v_t\}$ defined by the problem (2.2.1a–d) is *not* continuous $L_2(\Sigma) \rightarrow C([0, T]; H^2(\Omega) \times L_2(\Omega))$. Nevertheless, exact controllability/uniform stabilization results for the corresponding dissipative problem on such a space $H^2(\Omega) \times L_2(\Omega)$ of finite energy are given in [16, 17], with geometrical conditions relaxed or eliminated by virtue of the sharp trace results in [44].

**Example #3.** Another negative example where uniform stabilization is known, yet the operator $B^* L \in \mathcal{L}(L_2(0, T; U))$, is given by the Euler–Bernoulli plate equation with boundary control only in the 'moment' $\Delta w|_\Sigma$, as considered in [40, 14]. Here, the class of controls is $L_2(0, T; H^{\frac{1}{2}}(\Gamma))$, and the space of exact controllability and uniform stabilization is $Y = [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$. Exact controllability (without geometrical conditions) is established in [40], while uniform stabilization is proved in [40] (under geometrical conditions), and in [14] (without geometrical conditions). Optimal regularity of $L$ is given in [46, pp. 1023 and 1029]: it shows that it would take the class $H^{\frac{1}{2}, \frac{1}{2}}(\Sigma)$ of controls—thus with an extra $\frac{1}{2}$-derivative in time—to obtain $L$ continuous into $C([0, T]; [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega))$. Thus, by Theorem 2.1, $B^* L \notin \mathcal{L}(L_2(0, T]; H^{\frac{1}{2}}(\Gamma))$.

A further class of examples (Schrödinger equation with Neumann boundary controls) is deferred to Sect. 9.

# 3 Closed-Loop Nonlinear Feedback System: Uniform Stabilization with Optimal Decay Rates

We return to the closed-loop nonlinear version (1.3.15),

$$y_t = Ay - Bf(B^*y), \quad y(0) = y_0 \in Y, \tag{3.1}$$

under the same assumptions (a.1) for $A$ and (a.2) for $B$ of Subsect. 1.2 (which also covers the second order setting of Subsect. 1.3 under assumptions (h.1) and (h.2).

**Preliminary assumptions (for well-posedness).**

(H.1) The operator $A$ satisfies assumption (a.1) of Subsect. 1.3; the operator $B$ satisfies assumption (a.2) of Subsect. 1.3.

(H.2) The function $f$, $f(0) = 0$, is continuous $U \to U$ and monotone increasing

$$(f(u_1) - f(u_2), u_1 - u_2)_U \geqslant 0 \quad \forall\ u_1, u_2 \in U.$$

Under these assumptions, as noted below (1.3.15), it is known [19, 21] that $A - Bf(B^*)$ *generates a nonlinear semigroup of contractions*—say $S_F(t)$—which yields the following variation of parameter formula (3.1):

$$y(t) = S_F(t)y_0 = e^{At}y_0 - \{L(f(B^*S_F(\,\cdot\,)y_0)\}(t), \tag{3.2}$$

and obeys the energy identity

$$\|y(T)\|_Y^2 = \|y_0\|_Y^2 - 2\int_0^T (f(B^*y), B^*y)_U dt. \tag{3.3}$$

**Additional assumptions (for uniform stabilization).**

(H.3) The function $f$ satisfies $f(0) = 0$ and there exists a (real-valued) continuous, concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$, strictly increasing, with $h(0) = 0$, such that

$$\|f(u)\|_U^2 + \|u\|_U^2 \leqslant h((f(u), u)_U) \quad \forall\ u \in U. \tag{3.4}$$

See the end of the present Sect. 3: in the case of a substitution (Nemytski) operator, (H.3) is a *property*, not an assumption.

We next rescale $h$ by setting

$$\widetilde{h}(\,\cdot\,) = h\left(\frac{\cdot}{T}\right). \tag{3.5}$$

In order to state our main result on stabilizations, we need to introduce some functions. As in [24, 51], we set (with $C_T = \max\{2, 4k_T\}$, with $k_T = |||B^*L|||$; see (3.11) below)

$$H(x) = x + \frac{T}{2} C_T h\left(\frac{x}{T}\right):$$
$$\text{positive for } x > 0, \text{continuous, strictly increasing}, \quad H(0) = 0; \tag{3.6}$$

$$p(x) = H^{-1}(x):$$
$$\text{positive for } x > 0, \text{ continuous, strictly increasing}, \quad p(0) = 0; \tag{3.7}$$

$$q(x) = x - (I + p)^{-1}(x) = p(I + p)^{-1}(x) = (I + p)^{-1} p(x):$$
$$\text{positive for } x > 0, \text{ continuous, strictly increasing}, \quad q(0) = 0. \tag{3.8}$$

In particular, $H$, $p$, $q$, do *not* depend on the initial energy $\|u_0\|_Y$. We can now state the main uniform stabilization result of the present paper in the direction of [24, 25, 51]. Subsequent sections will provide a set of several PDE illustrations.

**Theorem 3.1.** *With reference to the nonlinear problem* (3.1), *assume the structural hypotheses* (H.1), (H.2) *on* $\{A, B, f\}$ *for well-posedness, as well as* (H.3). *Moreover, assume that*

(H.4) *the linear open-loop problem* (1.1.1) (= (2.1) *is exactly controllable on the state space* $Y$, *over the interval* $[0, T]$, $0 < T < \infty$, *within the class of* $L_2(0, T; U)$*-controls* $u$;

(H.5) *the open-loop, boundary* $\rightarrow$ *boundary map* $B^*L$ *is continuous* (*bounded*) *on* $L_2(0, T; U)$.

*Then the semigroup solution* $S_F(\,\cdot\,)$ *in* (3.2) *describing the solution of the closed-loop dissipative nonlinear problem* (3.1) (*as guaranteed by [19, 21] by virtue of* (H.1), (H.2)) *decays to zero on the space* $Y$ *as* $T \to +\infty$ *uniformly with respect to all initial data* $y_0$ *in* $Y$. *More precisely, its decay rate is described by the following nonlinear* ODE *in the scalar function* $s(t)$ (*nonlinear contraction*)

$$\frac{d}{dt}\, s(t) + q(s(t)) = 0, \qquad s(0) = \|y(0)\|_Y, \tag{3.9}$$

*where $q$ is defined in* (3.8)*, and hence does not depend on the initial energy $\|y_0\|_Y$. This means that the solutions $S_F(t)y_0$ of* (3.1) *satisfy*

$$\|S_F(t)y_0\|_Y \leqslant s(t)(\|y_0\|_Y) \searrow 0 \quad \text{as } t \nearrow +\infty, \tag{3.10}$$

*uniformly in $y_0 \in Y$. [Paper [24, Theorem 2, p. 511] also provides uniform decay rates in the presence of only the boundary damping]*

*Proof.* We start as in the proof of Proposition 1.3.2, and then modify the argument as to fall in the original treatment of [24], refined in [51].

*Step* 1. We apply $B^*$ on both sides of the solution formula (3.2), and obtain for $y_0 \in Y$:

$$B^* e^{At} y_0 = B^* S_F(t) y_0 + \{[B^* L] f(B^* S_F(\,\cdot\,)y_0)\}(t). \tag{3.11}$$

We next invoke assumption (H.5): Setting $k_T = |||B^*L|||$ in the uniform operator norm of $\mathcal{L}(L_2(0,T;U))$, we estimate from (3.11),

$$\|B^* e^{A\,\cdot}\, y_0\|^2_{L_2(0,T;U)}$$

$$\leqslant 2\|B^* S_F(\,\cdot\,)y_0\|^2_{L_2(0,T;U)} + 2k_T^2 \|f(B^* S_F(\,\cdot\,)y_0)\|^2_{L_2(0,T;U)} \tag{3.12}$$

$$\leqslant C_T \int_0^T \left[\|B^* S_F(t)y_0\|_U^2 + \|f(B^* S_F(t)y_0)\|_U^2\right] dt \tag{3.13}$$

$$\text{(by (3.4))} \quad \leqslant C_T \int_0^T h\left((f(B^* S_F(t)y_0), B^* S_F(t)y_0\right)_U dt \tag{3.14}$$

$$\leqslant T C_T h \left( \frac{1}{T} \int_0^T (f(B^* S_F(t)y_0), B^* S_F(t)y_0)_U dt \right). \tag{3.15}$$

In going from (3.14) to (3.15) we have invoked the Jensen inequality [18, p. 38]. Thus, recalling (3.5), we rewrite (3.15) as

$$\int_0^T \|B^* e^{At} y_0\|_U^2 dt \leqslant T C_T \widetilde{h} \left( \int_0^T (f(B^* S_F(t)y_0), B^* S_F(t)y_0)_U dt \right), \quad y_0 \in Y. \tag{3.16}$$

*Step* 2. The exact controllability assumption on the open-loop problem (1.2.1), in short, on the pair $\{A, B\}$, equivalently on the pair $\{A^*, B\}$, guarantees characterization (1.3.13). This, combined with (3.16), yields then, for any $y_0 \in Y$:

$$\begin{aligned}\|y_0\|_Y^2 &\leqslant C_T \int_0^T \|B^* e^{At} y_0\|_U^2 dt \\ &\leqslant T C_T \widetilde{h} \left( \int_0^T (f(B^* S_F(t) y_0), B^* S_F(t) y_0)_U dt \right). \qquad (3.17)\end{aligned}$$

*Step* 3. From the energy inequality (3.3) with $y(t) = S_F(t)y_0$, we have by virtue of the estimate (3.17),

$$\begin{aligned}&\|S_F(T)y_0\|_Y^2 \\ &\leqslant \|y_0\|_Y^2 + 2 \int_0^T (f(B^* S_F(t) y_0), B^* S_F(t) y_0)_U dt \qquad (3.18) \\ \text{(by (3.17))} \quad &\leqslant [2I + T C_T \widetilde{h}(\,\cdot\,)] \left( \int_0^T (f(B^* S_F(t) y_0), B^* S_F(t) y_0 \right)_U dt \qquad (3.19) \\ \text{(by (3.3))} \quad &\leqslant \left[ I + \frac{T}{2} C_T \widetilde{h}(\,\cdot\,) \right] \left[ \|y_0\|_Y^2 - \|S_F(T)y_0\|_Y^2 \right] \qquad (3.20) \\ &= H \left[ \|y_0\|_Y^2 - \|S_F(T)y_0\|_Y^2 \right], \qquad (3.21)\end{aligned}$$

with the map $H$ defined in (3.6). Thus, as the map $H$ is invertible on $\mathbb{R}^+$, from (3.21) we obtain

$$H^{-1}(\|S_F(T)y_0\|_Y^2) \leqslant \|y_0\|_Y^2 - \|S_F(T)y_0\|_Y^2, \qquad (3.22)$$

or with $y(T) = S_F(T)y_0$:

$$\|y(T)\|_Y^2 + H^{-1}(\|y(T)\|_Y^2) \leqslant \|y_0\|_Y^2. \qquad (3.23)$$

*Step* 4. To the inequality (3.24), we can now apply [24, Lemma 5.1], with $p = H^{-1}$ and $q$ as (3.7), (3.8), to obtain (3.9), (3.10), as desired. □

**In the case of a Nemytski operator, (H.3) is a property, not an assumption.**

In the case where the function $f$ is a Nemytski operator (operator of substitution), assumption (H.3) automatically follows from monotonicity of the feedback $f$ and the imposed growth condition. Thus, in this case, (H.3) is really a property, *not* an assumption.

To be more specific, we apply assumption (H.3) with

$$U = L_2(\Gamma), \; f : U \to U \text{ given by } f(u)(x) = g(u(x)), \tag{3.24}$$

where the scalar function $g(s)$ satisfies the following two conditions:

(i) $g \in C(\mathbb{R}), \; g(0) = 0, \; g$ is monotone increasing;

(ii)
$$m|s|^2 \leqslant g(s)s \leqslant Ms^2, \qquad \text{for } |s| \geqslant 1. \tag{3.25}$$

**Lemma 3.1.** *Under assumptions* (i) *and* (ii) *above, the function* $f : U \to U$ *satisfies hypothesis* (H.3).

*Proof.* Since $g$ is monotone increasing and satisfies (i), we have [24, 25, 51]: there exists a function $h_0$ monotone, $h_0(0) = 0$, concave such that [24, 25, 51]

$$s^2 + g^2(s) \leqslant h_0(sg(s)), \qquad |s| \leqslant 1. \tag{3.26}$$

We next claim that the function $h$ in assumption (H.3) is given by

$$h(s) \equiv (\text{meas } \Gamma) h_0 \left( \frac{s}{\text{meas } \Gamma} \right) + \left( \frac{1 + M^2}{m} \right) s, \tag{3.27}$$

where $h_0$ is a function constructed below from $g$. This follows by direct computations. In fact ([24, 25, 51]),

$$\|f(u)\|_U^2 + \|u\|_U^2 = \int_\Gamma [u^2 + |f(u)|^2] d\Gamma = \int_\Gamma [u^2(x) + |g(u(x)|^2] d\Gamma. \tag{3.28}$$

We split the last integral into two complementary parts. By use of both inequalities in (3.25), we have:

$$\int_{\{x \in \Gamma : u^2(x) \geqslant 1\}} [u^2(x) + |g(u(x))|^2] dx \leqslant \int_\Gamma (1 + M^2) |u(x)| \; |u(x)| d\Gamma \tag{3.29}$$

$$\leqslant \frac{1 + M^2}{m} \int_\Gamma u(x) g(u(x)) d\Gamma. \tag{3.30}$$

By (i), there exists (and is computable [24, 51]) a continuous, monotone increasing, concave function $h_0$, $h_0(0) = 0$, such that

$$s^2 + g^2(s) \leqslant h_0(sg(s)), \quad |s| \leqslant 1. \tag{3.31}$$

Hence, by (3.31),

$$\int_{\{x\in\Gamma : u^2(x)\geqslant 1\}} [u^2(x) + |g(u(x))|^2]dx \leqslant \int_{\Gamma} h_0(u(x)g(u(x))d\Gamma \tag{3.32}$$

$$\leqslant (\text{meas } \Gamma) h_0 \left( \frac{1}{\text{meas } \Gamma} \int_{\Gamma} u(x)g(u(x))d\Gamma \right), \tag{3.33}$$

where, in the last step, we have again invoked the Jensen inequality [18, p. 38]. Combining (3.30) and (3.33) on the RHS of (3.28) yields

$$\|f(u)\|_U^2 + \|u\|_U^2 \leqslant \left[ (\text{meas } \Gamma) h_0 \left( \frac{\cdot}{\text{mes } \Gamma} \right) + \frac{1 + M^2}{m} \right] \int_{\Gamma} u(x)g(u(x))d\Gamma, \tag{3.34}$$

and (3.4) of assumption (H.3) is verified, via (3.24), with the function $f$ defined in (3.27). □

# 4 A Second Order in Time Hyperbolic Illustration: The Wave Equation with Dirichlet Boundary Control and Suitably Lifted Velocity Boundary Observation

## *4.1 From the Dirichlet boundary control $g$ for the wave solution $\{v, v_t\}$ to the boundary observation $\frac{\partial z}{\partial \nu}|_{\Gamma}$, via the Poisson equation lifting $z = \mathcal{A}^{-1} v_t$*

Let $\mathbb{A}$ be the differential expression defined in (1.1.0).

**Linear open-loop and nonlinear closed-loop dissipative systems.**

In this subsection, let $\Omega$ be an open bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with sufficiently smooth boundary $\Gamma$. We consider the open-loop linear wave equation on $\Omega$ (with variable coefficients) with Dirichlet boundary control $g \in L_2(0, T; L_2(\Gamma)) \equiv L_2(\Sigma)$, and its corresponding nonlinear closed-loop boundary dissipative system:

$$\begin{cases} v_{tt} = \mathbb{A}v; \\ v(0,\,\cdot\,) = v_0, v_t(0,\,\cdot\,) = v_1; \\ v|_\Sigma = g; \end{cases} \begin{cases} w_{tt} = \mathbb{A}w & \text{in } Q; \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; \\ w|_\Sigma = f\left[\left.\dfrac{\partial(\mathcal{A}^{-1}w_t)}{\partial\nu}\right|_\Gamma\right] & \text{in } \Sigma, \end{cases} \quad \begin{matrix}(4.1.1a)\\(4.1.1b)\\(4.1.1c)\end{matrix}$$

with $Q = (0,T] \times \Omega$; $\Sigma = (0,T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined by (4.1.6) below: $\mathcal{A}\psi = -\mathbb{A}\psi$, $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$. Here, $\frac{\partial}{\partial\nu}$ denotes the co-normal derivative w.r.t. $\mathbb{A}$. The nonlinear function $f$ will be specified in Subsect. 4.4 below.

**Regularity, exact controllability of the $v$-problem; uniform stabilization of the $w$-problem for $f \equiv$ identity.**

References for this subsection include [3, 12, 26, 27, 29, 42, 63, 64, 22, 56, 91, 95, 53].

We begin by introducing the (state) space of optimal regularity

$$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv L_2(\Omega) \times H^{-1}(\Omega). \tag{4.1.2}$$

**Theorem 4.1.1** (regularity [26, 27, 22]). *Regarding the $v$-problem* (4.1.1), *with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds for each $T > 0$* (*the definition of $L$ given here is i n line with the abstract definition of the operator $L$ throughout this paper*) : *the map*

$$L : g \to Lg \equiv \{v, v_t\} \text{ is continuous} \\ L_2(\Sigma) \to C([0,T]; Y \equiv L_2(\Omega) \times H^{-1}(\Omega)). \tag{4.1.3}$$

**Theorem 4.1.2** (exact controllability [12, 29, 81, 64, 91, 95, 37, 50, 53]). *Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$ sufficiently large, there exists a $g \in L_2(\Sigma)$ such that the corresponding solution of the $v$-problem* (4.1.1) *satisfies $\{v(T), v_t(T)\} = 0$.*

**Theorem 4.1.3** (uniform stabilization [29, 41, 91]). *With reference to the $w$-problem* (4.1.1), *with $f(u) = u$, $u \in \Gamma$ (identity), we have*

(i) *the map $\{w_0, w_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup $e^{At}$ on $Y$;*

(ii)

$$w|_\Sigma = \frac{\partial(\mathcal{A}^{-1}w_t)}{\partial\nu} \in L_2(0,\infty; L_2(\Gamma)) \tag{4.1.4}$$

*continuously in* $\{w_0, w_1\} \in Y$;

(iii) *there exist constants* $M \geqslant 1$ *and* $\delta > 0$ *such that*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y \leqslant M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, \quad t \geqslant 0. \tag{4.1.5}$$

All three theorems above are obtained by PDE hard analysis energy methods (suitable energy multipliers). As usual, the most challenging result to prove is Theorem 4.1.3 on uniform stabilization: this, in addition, requires a shift of topology from $L_2(\Omega) \times H^{-1}(\Omega)$ (the space of the final result) to $H_0^1(\Omega) \times L_2(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable that is noted below in (4.1.10).

### Abstract model of $v$-problem.

With reference to (1.1.0), we let

$$\begin{aligned} &\mathcal{A}f = -\mathbb{A}f, \ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega); \ D : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \ s \in \mathbb{R}, \\ &\varphi = Dg \Longleftrightarrow \{\mathbb{A}\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g \text{ in } \Gamma\}. \end{aligned} \tag{4.1.6}$$

The abstract model for the $v$-problem in (4.1.1) is [29, 26, 27, 80]

$$v_{tt} = -\mathcal{A}v + \mathcal{A}Dg; \qquad \frac{d}{dt} \begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{4.1.7a}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \quad Bg = \begin{bmatrix} 0 \\ \mathcal{A}Dg \end{bmatrix}, \ B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = D^* x_2, \tag{4.1.7b}$$

where $*$ for $B$ and $D$ refer to different topologies, and where the Dirichlet map $D$ is defined in (4.1.6). Moreover, with $B^*$ defined by $(Bg, x)_Y = (g, B^*x)_{L_2(\Gamma)}$, with respect to the $Y$-topology in (4.1.2), we readily find the expression in (4.1.7).

### A 'dissipative-like,' open-loop, boundary control → boundary observation linear problem. The operator $B^*L$.

Given the $v$-problem in (4.1.1a-b-c) (LHS) with open-loop Dirichlet-control $g$, the argument below introduces a lifting of the velocity $v_t : z \equiv \mathcal{A}^{-1}v_t$. Indeed, with $y_0 = \{v_0, v_1\} = 0$, we show below that

$$B^* L g = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = D^* v_t(t; y_0 = 0) = D^* \mathcal{A}\mathcal{A}^{-1} v_t(t; y_0 = 0) \tag{4.1.8}$$

$$= -\frac{\partial}{\partial \nu} \mathcal{A}^{-1} v_t(t; y_0 = 0) = -\frac{\partial z(t)}{\partial \nu}; \tag{4.1.9}$$

$$z(t) \equiv \mathcal{A}^{-1} v_t(t; y_0 = 0) \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^1(\Omega)) \quad \text{continuously in } g \in L_2(\Sigma). \tag{4.1.10}$$

Indeed, to obtain (4.1.8), (4.1.9), one uses the definition of $L$ in (4.1.3) followed by the definition of $B^*$ in (4.1.7) and the usual property $D^* \mathcal{A} = -\frac{\partial}{\partial \nu}$ on $H_0^1(\Omega)$ [29, Equation (1.10)]. Finally, the regularity of $z$ in (4.1.10) follows from the regularity (4.1.3) on $v_t$ with $H^{-1}(\Omega) = [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. The new variable $z(t)$ defined in (4.1.10) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} = -\mathcal{A} z + D g_t \qquad \begin{cases} z_{tt} = \mathbb{A} z + D g_t & \text{in } Q; & (4.1.11\text{a}) \\ z(0, \cdot\,) = 0, \;\; z_t(0, \cdot\,) = z_1 & \text{in } \Omega; & (4.1.11\text{b}) \\ z|_\Sigma \equiv 0 & \text{in } \Sigma. & (4.1.11\text{c}) \end{cases}$$

Indeed, the abstract $z$-equation in (4.1.11) (left) is readily obtained from the abstract $v$-equation in (4.1.7), after applying throughout $\mathcal{A}^{-1}$ and $\frac{d}{dt}$ to it, and using the definition of $z(t)$ in (4.1.10). Moreover, since $z(t) \in H_0^1(\Omega)$ from (4.1.10), then $z$ satisfies the Dirichlet boundary condition in (4.1.11c). Moreover, in addition to the *a priori* regularity for $z$ in (4.1.10), we also have that for $z_t$:

$$z_t = \mathcal{A}^{-1} v_{tt} = \mathcal{A}^{-1}[-\mathcal{A} v + \mathcal{A} D g] = -v + D g \in L_2(0,T; L_2(\Omega)) \quad \text{continuously in } g \in L_2(\Sigma), \tag{4.1.12}$$

as it follows from $v \in C([0,T]; L_2(\Omega))$ by (4.1.3) and $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$ by (4.1.6) with $s = 0$. We next provide an interpretation of the new variable $z$ via the Poisson equation. From (4.1.10) we have

$$\mathcal{A} z = v_t(t; y_0 = 0); \;\text{ or } \begin{cases} \mathbb{A} z = -v_t(t, x; y_0 = 0) \text{ in } \Omega; & (4.1.13\text{a}) \\ z|_\Gamma = 0 \quad \text{on } \Gamma. & (4.1.13\text{b}) \end{cases}$$

$$z(t; x_0) = -\frac{1}{2\pi} \int_\Omega G(x, t; x_0) v_t(t, x; y_0 = 0) dx, \tag{4.1.14}$$

$G(\quad)$ being the associated Green function on $\Omega$ (with $t$ a parameter) [11]. Thus, $z$ is the solution of the corresponding Poisson equation with zero Dirichlet boundary data and with $-v_t$ as a forcing term.

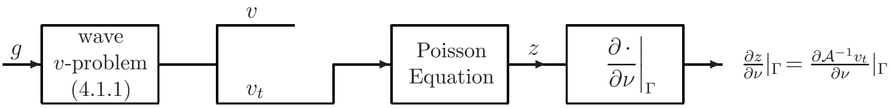


**Fig. 2** Open-loop boundary control $g \to$ boundary observation $\frac{\partial z}{\partial\nu}|_\Gamma$.

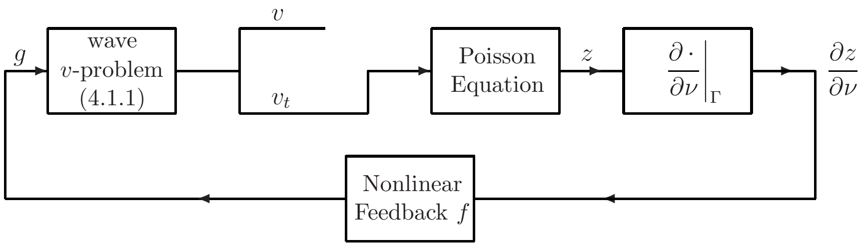


**Fig. 3** The corresponding closed-loop boundary dissipative $\{w, w_t\}$.

**Key boundary $\to$ boundary regularity question.**

With the optimal regularity of the variable $z$ given by (4.1.10), we consider the corresponding Neumann trace (boundary observation) and ask the question (recalling (4.1.8)–(4.1.9)):

$$\begin{aligned}&\text{Does } \frac{\partial z}{\partial\nu}\Big|_\Gamma \in L_2(0,T;L_2(\Gamma))?\\&i.e.,\\&\text{Is } B^*L \text{ continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma))?\end{aligned} \tag{4.1.15}$$

The question is not trivial when dim $\Omega \geqslant 2$. For dim $\Omega = 1$ it holds readily (see [48]).

In general, a positive answer to question (4.1.15) does *not* follow directly by trace theory from the optimal interior regularity (4.1.10) of $z$. In fact, a positive answer to question (4.1.15) would correspond to a "$\frac{1}{2}$ gain" in Sobolev-space regularity (in the space variable) over a *formal* application of trace theory to (4.1.10).

It turns out that the answer to question (4.1.15) is in the affirmative [49]. However, this sharp result does not follow entirely from the wealth of optimal regularity results and techniques for Mixed Problems (Initial Boundary Value Problems) for second order hyperbolic equations with Dirichlet homogeneous and nonhomogeneous boundary conditions, as given in [22, 26, 27], even though reference [22] provides a critical part of the proof (see Remark 4.1.1 below). The remaining part of the proof is also critical and challenging and is obtained by using a pseudodifferential analysis in the corresponding "elliptic sector" in the dual (Fourier) variables.

**Main result.**

The main result of the present Subsect. 4.1 is the following:

**Theorem 4.1.4.** *Let $\Omega$ be a sufficiently smooth bounded domain in $\mathbb{R}^n, n \geqslant 2$. We consider the $v$-problem in* (4.1.1a–c) (LHS)*, and zero initial conditions*: $v(0,\cdot) = v_t(0,\cdot) = 0$ *on $\Omega$. Then the regularity in* (4.1.15) *holds. This is to say, the open-loop map*

$$g \to B^*Lg = D^*v_t = -\frac{\partial z}{\partial \nu} \text{ is bounded on } L_2(\Sigma), \quad z = \mathcal{A}^{-1}v_t. \tag{4.1.16}$$

*Remark 4.1.1.* We note the marked contrast between the seriously challenging result (4.1.15) or (4.1.16) of the *open-loop* map $g \to -\frac{\partial \mathcal{A}^{-1}v_t}{\partial \nu}|_\Gamma$ of the $v$-problem (4.1.1a–c) on the one hand, and on the other hand, the much easier counterpart result of the corresponding *closed-loop* map given by (4.1.14) for the closed-loop boundary dissipative $w$-problem in (4.1.1a–c) (RHS) with $f =$ identity, which may be thought of as being obtained from the $v$-problem by closing up the loop, whereby the output $-\frac{\partial \mathcal{A}^{-1}v_t}{\partial \nu}$ is required to coincide with the input $g$. (Compare Figs. 2 and 3.) □

**The energy method on the mixed PDE $z$-problem (4.1.11) fails to show that $\frac{\partial z}{\partial \nu} \in L_2(0,T;L_2(\Gamma))$, continuously in $g \in L_2(0,T;L_2(\Gamma))$, except in the 1-dimensional case.**

To make our point, it will suffice to consider the case $\mathbb{A} = \Delta$. See the end of Subsect. 1.2. As in [22], multiplying the PDE problem (4.1.11) by $h \cdot \nabla z$, with $h$ a $C^2$-vector field on $\overline{\Omega}$, with $h|_\Gamma = \nu$ on $\Gamma$, and using the boundary condition (4.1.11c), we obtain the identity [22, Equation (2.27), p. 157]

$$\frac{1}{2}\int_{\Sigma}(T-t)\left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma$$

$$= \int_{Q}(T-t)H\nabla z\cdot\nabla z\,dQ + \frac{1}{2}\int_{Q}(T-t)[z_t^2-|\nabla z|^2]\text{div } h\,dQ$$

$$+\int_{Q} z_t h\cdot\nabla z\,dQ - \int_{Q}(T-t)Dg_t h\cdot\nabla z\,dQ. \tag{4.1.17}$$

[Since $z_t$ is only $L_2$ in time (see (4.1.12)), we have used the multiplier $(T-t)h\cdot\nabla z$, to eliminate the terms at $t=0$ and $t=T$. Otherwise, one takes preliminarily $g$ in the class (4.1.19) below, and uses just the multiplier $h\cdot\nabla z$.]

Thus, the *a priori* regularity of $\{z, z_t\}$ in (4.1.10) and (4.1.12) guarantee that all first three integral terms on the RHS of (4.1.12) are well defined, continuously in $g\in L_2(\Sigma)$. Hence from (4.1.12) we obtain

$$\frac{1}{2}\int_{\Sigma}(T-t)\left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right) - \int_{Q}(T-t)Dg_t\,h\cdot\nabla z\,dQ. \tag{4.1.18}$$

Letting now $g$ be (temporarily) in the class

$$g\in C([0,T];L_2(\Gamma)) \qquad g(T)=g(0)=0, \tag{4.1.19}$$

dense in $L_2(\Sigma)$, we see by integration by parts in $t$ with use of (4.1.19), followed by the usual divergence theorem, that

$$-\int_{Q}(T-t)Dg_t\,h\cdot\nabla z\,dQ = \int_0^T\int_{\Omega} Dg\,h\cdot\nabla z_t\,d\Omega\,dt + l.o.t. \tag{4.1.20}$$

$$= \int_0^T\int_{\Gamma} Dg\,z_t h\cdot\nu\,d\Gamma\,dt - \int_0^T\int_{\Omega} z_t h\cdot\nabla(Dg)d\Omega\,dt$$

$$-\int_0^T\int_{\Omega} Dg\,z_t\text{div } h\,d\Omega\,dt + l.o.t., \tag{4.1.21}$$

in view of $z_t|_{\Gamma}=0$ by (4.1.11c). The last integral term in the RHS of (4.1.21) is well-defined continuously in $g\in L_2(\Sigma)$, by (4.1.12) on $z_t$ and $Dg\in L_2(0,T;H^{\frac{1}{2}}(\Omega))$. Thus, from (4.1.18) we obtain via (4.1.21)

$$\int_{\Sigma} \left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right) + \int_0^T \int_{\Omega} z_t h \cdot \nabla(Dg) d\Omega\, dt. \tag{4.1.22}$$

**One-dimensional case.**

In the one-dimensional case, say $\Omega = (0,1)$, with $\mathbb{A} = \Delta = \partial_{xx}$, and boundary conditions $v|_{x=0} = g$, $v|_{x=1} = 0$, we then have that $(Dg)(x)$ is a linear function of $x$: $(Dg)(x) = -gx + g$, $g \in \mathbb{R}$, $0 \leqslant x \leqslant 1$. Thus, we have $\nabla(Dg) \equiv -g$ (constant), and we get

$$\int_{\Sigma} \left(\frac{\partial z}{\partial \nu}\right)^2 d\Sigma = \mathcal{O}\left(\|g\|^2_{L_2(\Sigma)}\right), \tag{4.1.23}$$

thus re-proving— in a more complicated way!—the result of [26, Subsect. 4.7].

**Multidimensional case: dim $\boldsymbol{\Omega} \geqslant 2$.**

In this case, the *a priori* regularity of $z_t \in L_2(0,T; L_2(\Omega))$ and $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$, hence $|\nabla(Dg)| \in L_2(0,T; (H^{\frac{1}{2}}_{00}(\Omega))')$ [65, p. 85] show that, roughly speaking, "$\frac{1}{2}$" *space derivative is apparently missing* in order to have the integral term on the RHS of (4.1.22) well defined.

*Remark 4.1.2.* The above Theorem 4.1.4 was first stated in [1] (see the estimate (2.7) in p. 121). We believe that the proof that we give in Subsect. 4.2 below is essentially self-contained and much simpler than the sketch given in [1]. The idea pursued in [1] is based on a full microlocal analysis of the fourth order operator $\Delta(D_t^2 - \Delta)$ [where the extra $\Delta$ is used to eliminate $Dg$ from the $z$-dynamics $z_{tt} = \Delta z + Dg_t$ (see (4.1.11a)), as $\Delta Dg_t \equiv 0$]. The subsequent microlocal analysis of [1] considers, as usual [1], three regions: the hyperbolic region, the elliptic region, and the "glancing rays" region. The latter is the most demanding, and it is unfortunate that no details are provided in [1] for the analysis in the glancing region, except for reference to author's Ph.D. thesis.

By contrast, our proof in Subsect. 2.2 below [49] invokes, for the most critical part, the sharp regularity of the wave equation from [22]—which is obtained via differential, rather than pseudo-differential/micro-local analysis methods. In addition, standard elliptic (interior and) trace regularity of the Dirichlet map $D$ is used. Thus, by simply invoking these results in Equation (4.1.12) above for $z_t$, we obtain—by purely differential methods, the critical result on $\frac{\partial z_t}{\partial \nu}$ of Step 1, Equation (4.2.3). This then provides automatically the desired regularity of $\frac{\partial z}{\partial \nu}$ microlocally outside the elliptic sector of the

D'Alambertian $\Box = D_t^2 - \Delta$, where the time variable dominates the tangential space variable in the Fourier space (see (4.2.11) below).

Thus, the rest of the proof follows from PDO elliptic regularity of the localized problem. $\Box$

## 4.2 Proof of Theorem 4.1.4

*Step* 1. Let $g \in L_2(\Sigma)$. Then the following interior and boundary sharp regularity for the $v$-problem (4.1.1a–c) (LHS) is known [22, Theorem 2.3, p. 153; or else Theorem 3.3, p. 176 (interior regularity) plus Theorem 3.7, p. 178 (boundary regularity)]

$$\{v, v_t\} \in C([0,T]; L_2(\Omega) \times H^{-1}(\Omega)); \quad \frac{\partial}{\partial \nu} v \Big|_{\Sigma} \in H^{-1}(\Sigma) \tag{4.2.1}$$

continuously in $g$ (as noted in (4.1.3)). Moreover, elliptic regularity of the Dirichlet map gives $Dg \in L_2(0,T; H^{\frac{1}{2}}(\Omega))$, and thus

$$\frac{\partial}{\partial \nu} Dg \in L_2(0,T; H^{-1}(\Gamma)). \tag{4.2.2}$$

[This result can be proved by interpolation between

$$\begin{cases} \Delta h = 0 \text{ in } \Omega \\ h|_{\Gamma} = g \in H^{\frac{1}{2}}(\Gamma) \end{cases} \Rightarrow h \in L_2(\Omega) \text{ and } \frac{\partial h}{\partial \nu}\Big|_{\Gamma} \in H^{-\frac{3}{2}}(\Gamma),$$

and

$$\begin{cases} \Delta h = 0 \text{ in } \Omega \\ h|_{\Gamma} = g \in H^{\frac{1}{2}}(\Gamma) \end{cases} \Rightarrow h \in H^1(\Omega) \text{ and } \frac{\partial h}{\partial \nu}\Big|_{\Gamma} \in H^{-\frac{1}{2}}(\Gamma).\ ]$$

Next, using (4.2.1) and (4.2.2) in (4.1.12) yields

$$\frac{\partial}{\partial \nu} z_t = -\frac{\partial}{\partial \nu} v + \frac{\partial}{\partial \nu} Dg \in H^{-1}(\Sigma). \tag{4.2.3}$$

The above relation provides us with the desired regularity of $\frac{\partial z}{\partial \nu}$ microlocally outside the elliptic sector of the D'Alambertian $\Box = D_t^2 - \Delta$; i.e., when the dual Fourier variable $\sigma$ (corresponding to time) dominates the dual Fourier variable $|\eta|$ (corresponding to the space tangential variable). A quantitative statement of this is given in (4.2.11) below.

*Step* 2. It remains to show that the $L_2$ regularity of $\frac{\partial z}{\partial \nu}$ holds also in the elliptic sector. This is done by standard arguments using localization of the PDO symbols. We use standard partition of unity procedure and local change of coordinates by which $\Omega$ and $\Gamma$ can be identified (locally) with $\widetilde{\Omega} \equiv \{(x,y) \in \mathbb{R}^n,\ x \geqslant 0, y \in \mathbb{R}^{n-1}\}$, $\widetilde{\Gamma} \equiv \{(x,y) \in \mathbb{R}^n,\ x = 0, y \in \mathbb{R}^{n-1}\}$. The second order elliptic operator $\Delta$ is identified in local coordinates (Melrose-Sjostrand) with $\widetilde{\Delta} = D_x^2 + r(x,y)D_y^2 + lot$, where $lot$ (which result from commutators) are first order differential operators and $r(x,y)D_y^2$ stands for the second order tangential (in the $y$ variable) strongly elliptic operator. Since solutions $v$ satisfy zero initial data, we can also extend $v(t)$ by zero for $t < 0$. For $t > T$ we multiply the solution by a smooth cutoff function $\phi(t) = 0, t \geqslant \frac{3}{2}T, \phi(t) = 1, t \leqslant T$. Thus, in order to obtain the desired solution, it amounts to consider the following problem:

$$\begin{aligned} w_{tt} = \widetilde{\Delta} w = \Delta_0 w + lot(v) \text{ in } \widetilde{Q},\ w|_{\widetilde{\Gamma}} = g;\ w(0,\,\cdot\,) \\ = w_t(0,\,\cdot\,) = 0 \text{ in } \widetilde{\Omega},\ \operatorname{supp} w \in [0, 2T] \end{aligned} \tag{4.2.4a}$$

[the solution $w$ in (4.2.4a) should not be confused with the solution $w$ of the closed-loop dissipative problem in (4.1.1a–c) (RHS)]. Here, $\Delta_0 = D_x^2 + r(x,y)D_y^2$ is the principal part of $\widetilde{\Delta}$ and $v$ is the original solution $v = Lg$ of the $v$-problem on the LHS of (4.1.1a–c). Below, we write $w = u + y$, where $u, y$ satisfy (4.2.5) and (4.2.6), respectively. As a consequence, we obtain

$$\{w, w_t\} \in C([0,T]; L_2(\widetilde{\Omega}) \times H^{-1}(\widetilde{\Omega})) \text{ continuously in } g \in L_2(\widetilde{\Sigma}). \tag{4.2.4b}$$

Here and below, we call $u$ the solution of

$$u_{tt} = \Delta_0 u \text{ in } \widetilde{Q},\ u|_{\widetilde{\Sigma}} = g;\ u(0,\,\cdot\,) = u_t(0,\,\cdot\,) = 0 \text{ in } \widetilde{\Omega}, \tag{4.2.5a}$$

$$\{u, u_t\} \in C([0,T]; L_2(\widetilde{\Omega}) \times H^{-1}(\widetilde{\Omega})) \text{ continuously in } g \in L_2(\widetilde{\Sigma}), \tag{4.2.5b}$$

the counterpart regularity statement of (4.2.1) for $v$ in $\Omega$. Likewise, we introduce the following nonhomogenous problem:

$$y_{tt} = \Delta_0 y + f \text{ in } \widetilde{Q},\ y|_{\widetilde{\Sigma}} = 0,\ y(0,\,\cdot\,) = y_t(0,\,\cdot\,) = 0 \text{ in } \widetilde{\Omega}, \tag{4.2.6}$$

where $f = lot\ (v)$ results from the presence of the lower order terms applied to the original variable $v$ in (4.2.1). Thus, recalling that $v \in C([0,T]; L_2(\Omega))$ by (4.2.1), we obtain

$$f \in C([0,T]; H^{-1}(\widetilde{\Omega})), \text{ hence } \{y, y_t\} \in C([0,T]; L_2(\widetilde{\Omega}) \times H^{-1}(\widetilde{\Omega})) \tag{4.2.7}$$

[22, Theorem 2.3, p. 153] continuously in $g \in L_2(\Sigma)$.

By the principle of superposition, we have $w = u + y$, as announced above.

*Step* 3. In this step, we handle the $y$-problem (4.2.6). We first recall from (4.1.16) that our original objective is showing that $D^* v_t \in L_2(\Sigma)$ continuously in $g \in L_2(\Sigma)$. Moreover, we recall that $v$ in $\Omega$ is transferred into $w = u + y$, on the half-space $\widetilde{\Omega}$ (locally). Thus, by (4.2.6), (4.2.7), what suffices to show for $y$ is the following regularity property

$$f \to D^* y_t : \text{ continuous } L_2(0,T; H^{-1}(\widetilde{\Omega})) \to L_2(0,T; L_2(\widetilde{\Gamma})), \qquad (4.2.8)$$

whereby $D^* y_t$ is ultimately continuous in $g \in L_2(\Sigma)$. However, the above property (4.2.8) is known from [22, Theorem 3.11, p. 182] and has been used in the past several times. In fact, set $A = -\Delta_0$, with $\mathcal{D}(A) = H^2(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$ and rewrite (4.2.6) abstractly as: $y_{tt} = -Ay + f$. Apply $A^{-1}$ throughout and set $\Psi = A^{-1} y \in C([0,T]; \mathcal{D}(A))$ via (4.2.7). Moreover, $A^{-1} f \in L_2(0,T; H_0^1(\widetilde{\Omega}))$, again by (4.2.7). Thus, $\Psi$ solves the problem

$$\Psi_{tt} = \Delta_0 \Psi + A^{-1} f \text{ in } \widetilde{Q}, \; \Psi|_{\widetilde{\Sigma}} = 0, \; \Psi(0, \cdot\,) = \Psi_t(0, \cdot\,) = 0 \text{ in } \widetilde{\Omega}. \qquad (4.2.9)$$

We further have that $A^{-1} y_t \in C([0,T]; H_0^1(\widetilde{\Omega}))$, again by (4.2.7). Finally we recall that $D^* A A^{-1} y_t = -\frac{\partial}{\partial \nu} \Psi_t$ [46, (4.1.9)]. One can simply quote [22, Theorem 3.11, p. 182] or [46, Equation (10.5.5.11), p. 952] to obtain the desired regularity (4.2.8):

$$D^* y_t = -\frac{\partial}{\partial \nu} \Psi_t \in L_2(\widetilde{\Sigma}), \text{ continuously in } g \in L_2(\Sigma). \qquad (4.2.10)$$

*Step* 4. Having accounted for the $lot(v)$ in Step 3—which are responsible for the $y$-problem—we may in this step set $y \equiv 0$ and thus identify $w$ with $u : w \equiv u$. Thus, it remains to consider the problem (4.2.5) in $u$, involving only the principal part of the D'Alambertian. Let $\mathcal{X} \in S^0(\widetilde{Q})$ denote the PDO operator $\mathcal{X}(x,y,t)$ with smooth symbol of localization $\chi(x,y,t,\sigma,\eta)$ supported in the elliptic sector of $\Box \equiv D_t^2 - D_x^2 - r(x,y) D_y^2$, where the principal part of the D'Alambertian is written in local coordinates. The dual variables $\sigma \in \mathbb{R}^1$, $\eta \in \mathbb{R}^{n-1}$ correspond to the Fourier variables of $t \to i\sigma$, $y \to i\eta$. Thus, supp $\chi \in \{(x,y,t,\sigma,\eta) \in \widetilde{Q} \times \mathbb{R}^1 \times \mathbb{R}^{n-1}, \sigma^2 - r(0,y)|\eta|^2 < 0\}$. The established regularity (4.2.3) and the fact that $|\sigma| \geqslant c|\eta|$ on supp $\chi$ imply

$$(I - \mathcal{X}) \frac{\partial}{\partial \nu} z \in L_2(\Sigma), \qquad (4.2.11)$$

a statement that $|\sigma| \frac{\partial z}{\partial \nu}$, and thus *a fortiori* $|\eta| \frac{\partial z}{\partial \nu}$, are in $L_2$ in time and space in the (hyperbolic) sector $|\sigma| \geqslant c|\eta|$. On the other hand, returning to the problem (4.2.5) for $u$, rewritten as $\Box u = 0$ and applying $\mathcal{X}$, we see that the variable $\mathcal{X} u$ satisfies

$$\Box \mathcal{X} u = -[\mathcal{X}, \Box] u \in H^{-1}(\widetilde{Q}). \qquad (4.2.12)$$

where henceforth we take for $\widetilde{Q}$ an extended cylinder based on $\widetilde{\Omega} \times [-T, 2T]$. Indeed, this last inclusion follows from $[\mathcal{X}, \Box] \in S^1(\widetilde{Q})$ and the *priori* regularity (4.2.5b) for $u$ implying $u \in L_2(\widetilde{Q})$, which jointly lead to $[\mathcal{X}, \Box] u \in$

$H^{-1}(\widetilde{Q})$. Moreover, $\mathcal{X}u|_{\Gamma} = \mathcal{X}g \in L_2(\widetilde{\Sigma})$. Furthermore, still by (4.2.5b) and the fact that supp $u \in [0, \frac{3}{2}T]$ we have, by the pseudo-local property of pseudodifferential operators, that $(\mathcal{X}u)(2T) \in C^\infty(\widetilde{\Omega})$, $(\mathcal{X}u)(-T) \in C^\infty(\widetilde{\Omega})$. We conclude that $\mathcal{X}u|_{\partial\widetilde{Q}} \in L_2(\partial\widetilde{Q})$, a boundary condition to be associated to (4.2.12). Since $\Box\mathcal{X}$ is a pseudodifferential elliptic operator, classical elliptic theory, applied to

$$\Box\mathcal{X}u \in H^{-1}(\widetilde{Q}),\ \mathcal{X}u|_{\partial\widetilde{Q}} \in L_2(\partial\widetilde{Q})$$

—the elliptic problem obtained above—yields

$$\mathcal{X}u \in H^{\frac{1}{2}}(\widetilde{Q}) + H^1(\widetilde{Q}) \subset H^{\frac{1}{2}}(\widetilde{Q}), \tag{4.2.13}$$

where the first containment on the RHS of (4.2.13) is due to the boundary term, and the second to the interior term. Next, we return to the elliptic problem: $\mathbb{A}z = -v_t$ in $Q$, $z|_{\Sigma} = 0$ from (4.1.13), with *a priori* regularity noted in (4.1.10). The counterpart of the above elliptic problem in the half-space $\widetilde{Q}$ (locally) is: $\widetilde{\Delta}z = -u_t$ in $\widetilde{Q}$, $z|_{\widetilde{\Sigma}} = 0$ (we retain the symbol $z$ in $\widetilde{Q}$), as we are identifying $w$ with $u$ in the present Step 4 (due to the results of Step 3). Applying $\mathcal{X}$ throughout yields

$$\widetilde{\Delta}\mathcal{X}z = -\mathcal{X}u_t + [\widetilde{\Delta}, \mathcal{X}]z = -\frac{d}{dt}\mathcal{X}u + \left[\frac{d}{dt}, \mathcal{X}\right]u + [\widetilde{\Delta}, \mathcal{X}]z. \tag{4.2.14}$$

Note $[\widetilde{\Delta}, \mathcal{X}] \in S^1(\widetilde{Q})$ and $[\frac{d}{dt}, \mathcal{X}] \in S^0(\widetilde{Q})$. Hence, by the *a priori* regularity in (4.2.5b) for $u$ and in (4.1.13) for $z$, we conclude

$$\left[\frac{d}{dt}, \mathcal{X}\right]u + [\widetilde{\Delta}, \mathcal{X}]z \in L_2(\widetilde{Q}). \tag{4.2.15}$$

Moreover, by (4.2.13), $\frac{d}{dt}\mathcal{X}u \in H_{(0,-\frac{1}{2})}(\widetilde{Q})$ where we have used the anisotropic Hörmander spaces [13, Vol. III, p. 477], $H_{(m,s)}(\widetilde{Q})$, where $m$ is the order in the normal direction to the plane $x = 0$ (which plays a distinguished role) and $(m + s)$ is the order in the tangential direction in $t$ and $y$. Via (4.15), we are thus led to solving the problem

$$\widetilde{\Delta}\mathcal{X}z \in H_{(0,-\frac{1}{2})}(\widetilde{Q}) + L_2(\widetilde{Q}), \quad (\mathcal{X}z)|_{\widetilde{\Gamma}} = 0. \tag{4.2.16}$$

By elliptic regularity (note that $\widetilde{\Delta}\mathcal{X}$ is elliptic in $\widetilde{Q}$), we obtain again

$$\mathcal{X}z \in H^{\frac{3}{2}}(\widetilde{Q}), \quad \frac{\partial}{\partial\nu}\mathcal{X}z \in L_2(\widetilde{\Sigma}). \tag{4.2.17}$$

Combining (4.2.17) and (4.2.11) yields the final conclusion

$$\frac{\partial}{\partial \nu} z = (I - \mathcal{X}) \frac{\partial}{\partial \nu} z + \mathcal{X} \frac{\partial}{\partial \nu} z \in L_2(\widetilde{\Sigma}), \tag{4.2.18}$$

and Theorem 4.1.1 is proved. □

*Remark 4.2.1.* Having established Theorem 4.1.4, we may then return to identity (4.1.22). Since its LHS is finite by Theorem 4.1.4, we conclude that

$$\int_0^T \int_\Omega z_t h \cdot \nabla(Dg) d\Omega \, dt < \infty,$$

a result that is not apparent, save for the 1-dimensional case as in (4.1.23).

## *4.3 The half-space problem: A direct computation*

In this subsection, we consider the wave equation defined on a 2-dimensional half-space, with Dirichlet boundary control. So let

$$\begin{aligned} &\Omega \equiv \mathbb{R}_2^+ = \{(x, y) : x \geqslant 0, \ y \in \mathbb{R}^{n-1}\}, \\ &\Gamma = \{(0, y) : y \in \mathbb{R}^{n-1}\} = \Omega|_{x=0}. \end{aligned} \tag{4.3.1}$$

On $\Omega$ we consider the wave equation with Dirichlet boundary control:

$$\begin{cases} v_{tt} = v_{xx} + D_y^2 v & \text{in } Q \equiv (0, \infty] \times \Omega; & (4.3.2\text{a}) \\ v(0, \cdot) = 0, \ v_t(0, \cdot) = 0 & \text{in } \Omega; & (4.3.2\text{b}) \\ v|_\Sigma = g & \text{in } \Sigma \equiv (0, \infty) \times \Gamma, & (4.3.2\text{c}) \end{cases}$$

where $g \in L_2(0, \infty; L_2(\Gamma))$ and $D_y^2 v = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial y_j^2} v$, $y = [y_1, \ldots, y_{n-1}]$. We have seen in Subsect. 4.1, Equation (4.1.8), that for the problem (4.3.2) we have

$$B^* L g = D^* v_t. \tag{4.3.3}$$

**Theorem 4.3.1.** *With reference to the half-space problem* (4.3.2a–c), *we have in the notation of* (4.3.3), (5.1.8), (5.1.9):

$$\begin{aligned} &g \to B^* L g = D^* v_t(t; y_0 = 0) = -\left.\frac{\partial z}{\partial \nu}\right|_\Gamma \\ &\textit{is continuous on } L_2(\Sigma), \ \Sigma = (0, T) \times \Gamma. \end{aligned} \tag{4.3.4a}$$

*Proof of* (4.3.3). Our proof is inspired by [36, Counterexample, p. 294], for a different type of result.

*Goal*: Given $T > 0$ and $g \in L_2(0,T; L_2(\Gamma))$, we extend $g$ by zero for $t > T$. It will then suffice to show that

$$e^{-\gamma t}(B^* Lg)(t) \in L_2(0,\infty; L_2(\Gamma)), \tag{4.3.4b}$$

for a fixed constant $\gamma > 0$.

*Step* 1. Let $\widehat{v}(\tau, x, \eta)$ denote the Laplace–Fourier transform of $v(t, x, y)$: Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in \mathbb{R}$, and Fourier in $y \to i\eta$, $\eta \in \mathbb{R}^{n-1}$, leaving $x \geqslant 0$ as a parameter. We then obtain for the solution of (4.3.2) vanishing at $x = \infty$:

$$\tau^2 \widehat{v} = \widehat{v}_{xx} - |\eta|^2 \widehat{v}; \qquad \text{or } \widehat{v}(\tau, x, \eta) = \widehat{g}(\tau, \eta) e^{-\sqrt{\tau^2 + |\eta|^2} x}, \ x \geqslant 0;$$
$$\tau^2 + |\eta|^2 = (\gamma^2 + |\eta|^2 - \sigma^2) + 2i\gamma\sigma. \tag{4.3.5}$$

*Step* 2. Let $\varphi \in L_2(0,\infty; L_2(\Gamma))$. We consider the Laplace equation in $\Omega$, with Dirichlet boundary condition on $\Gamma$ given by $\varphi$ a.e. in $t$, i.e., in the notation for $D$ in (4.1.6):

$$u = D\varphi, \text{ where } u_{xx} + D_y^2 u = 0 \text{ in } \Omega; \ u|_\Gamma = \varphi \text{ in } \Gamma. \tag{4.3.6}$$

The solution $u = D\varphi$ of the problem (4.3.5) is given by the well-known formula in the transformed variables [11, Subsect. 9.7.3, p. 375]:

$$\widehat{u}(\tau, x, \eta) = \widehat{D\varphi}(\tau, x, \eta) = \widehat{\varphi}(\tau, \eta) e^{-|\eta| x} \ \forall \ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1}, \ x \geqslant 0. \tag{4.3.7}$$

*Step* 3. According to (4.3.4), it suffices to show that, for a fixed constant $\gamma > 0$, we have with $L_2(\Sigma_\infty) = L_2(0,\infty; L_2(\Gamma))$:

$$\big(e^{-2\gamma t} B^* Lg, u\big)_{L_2(\Sigma_\infty)} = \big(e^{-2\gamma t} D^* v_t(\,\cdot\,; y_0 = 0), u\big) < \infty \ \forall \ g, u \in L_2(\Sigma_\infty). \tag{4.3.8}$$

*Step* 3(i). First, we establish that: for all $g, u \in L_2(0,\infty; L_2(\Gamma)) = L_2(\Sigma_\infty)$, we have

$$\big(e^{-2\gamma t} B^* Lg, u\big)_{L_2(\Sigma_\infty)} = \frac{1}{2\pi} \int\limits_{\mathbb{R}^n_{\sigma,\eta}} \left( \tau \int\limits_0^\infty e^{-\sqrt{\tau^2 + |\eta|^2} x} e^{-|\eta| x} dx \right) \widehat{g}(\tau, \eta) \overline{\widehat{u}(\tau, \eta)} d\sigma \, d\eta, \tag{4.3.9}$$

where $\mathbb{R}^n_{\sigma\eta}$ denotes the $n$-dimensional Euclidean space in the variables $\sigma$ and $\eta \in \mathbb{R}^{n-1}$. □

*Proof of* (4.3.9). Recalling (4.3.3), the Parseval identity for Laplace transforms [6, Theorem 31.8, p. 212] and (4.3.5), (4.3.7), we compute ($\sim$ indicates the Laplace transform in (4.3.12)), where $\tau = \gamma + i\sigma$:

$$\big(e^{-2\gamma t}(B^*Lg)(t), u(t)\big)_{L_2(0,\infty;L_2(\Gamma))}$$

$$= \int_0^\infty e^{-2\gamma t}(B^*Lg, u)_{L_2(\Gamma)}dt \tag{4.3.10}$$

$$\text{(by (4.3.3))} = \int_0^\infty e^{-2\gamma t}(D^*v_t, u)_{L_2(\Gamma)}dt = \int_0^\infty e^{-2\gamma t}(v_t, Du)_{L_2(\Omega)}dt \tag{4.3.11}$$

$$\text{(by [6, p. 212])} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\widetilde{v}_t(\tau,x,y), \widetilde{Du}(\tau,x,y))_{L_2(\Omega)}d\sigma \tag{4.3.12}$$

$$= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^n_{\sigma\eta}}\int_0^\infty \tau\widehat{v}(\tau,x,\eta)\widehat{Du}(\tau,x,\eta)dx\, d\sigma\, d\eta \tag{4.3.13}$$

$$\text{(by (4.3.5),(4.3.7))} = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^n_{\sigma\eta}}\int_0^\infty \tau\widehat{g}(\tau,\eta)e^{-\sqrt{\tau^2+|\eta|^2}x}\ \overline{\widehat{u}(\tau,\eta)}e^{-|\eta|x}dx\, d\sigma\, d\eta \tag{4.3.14}$$

$$= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^n_{\sigma\eta}}\left(\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx\right)\widehat{g}(\tau,\eta)\overline{\widehat{u}(\tau,\eta)}d\sigma\, d\eta, \tag{4.3.15}$$

and (4.3.15) establishes (4.3.9), as desired. In (4.3.12), (4.3.13), we have invoked Parseval formula for Laplace $t \to \tau$ [6, p. 212] and Fourier transform $y \to i\eta$; while in (4.3.14), we have recalled (4.3.5) and (4.3.7) with $\varphi = g$.

*Step* 3(ii). With $\tau = \gamma + i\sigma$, let

$$H(\sigma,\eta) \equiv \sigma\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx. \tag{4.3.16}$$

It is immediate to show that $|H(\sigma,\eta)|$ is uniformly bounded for all $(\sigma,\eta)\in\mathbb{R}^n_{\sigma,\eta}$:

$$|H(\sigma,\eta)|\leqslant C<\infty,\ \text{for all } \sigma\in\mathbb{R}^1,\ \eta\in\mathbb{R}^{n-1}. \tag{4.3.17}$$

Indeed, set

$$A+iB\equiv\sqrt{\tau^2+|\eta|^2},\quad A^2-B^2=\gamma^2+|\eta|^2-\sigma^2,\ AB=2\gamma\sigma. \tag{4.3.18}$$

Then we first note from (4.3.18) that

$$|H(\sigma,\eta)|\equiv\left|\sigma\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx\right|$$

$$\leqslant c\frac{|\sigma|}{|A|+|\eta|+|B|}\equiv c\ h(\sigma,\eta). \tag{4.3.19}$$

In the elliptic region, say $\{|\sigma|\leqslant 2|\eta|,\ \sigma^2+|\eta|^2\geqslant 1\}$, we readily have from (4.3.19) that $h(\sigma,\eta)\leqslant\frac{|\sigma|}{|\eta|}\leqslant 2$. On the other hand, solving the system in (4.3.18) by elementary computations, we obtain

$$A^2=\frac{8\gamma^2\sigma^2}{\{(\sigma^2-|\eta|^2-\gamma^2)^2+16\gamma^2\sigma^2\}^{\frac{1}{2}}+(\sigma^2-|\eta|^2-\gamma^2)}$$

$$\sim\frac{|\,\sigma/\eta|^2}{|\,\sigma/\eta|^2}=1, \tag{4.3.20}$$

say for $\sigma^2+|\eta|^2\geqslant 1$, whereby then (4.3.18) gives: $B\sim\sigma$. Hence, by (4.3.19),

$$h(\sigma,\eta)\leqslant\frac{|\sigma|}{|B|}\sim\frac{|\sigma|}{|\sigma|}=1.$$

Thus,

$$|H(\sigma,\eta)|\leqslant C<\infty,\ \text{for all } \sigma\in\mathbb{R}^1,\ \eta\in\mathbb{R}^{n-1}. \tag{4.3.21}$$

Then (4.3.9) and (4.3.21) yield the desired conclusion:

$$|(e^{-2\gamma t}B^*Lg,u)_{L_2(\Sigma_\infty)}|\leqslant C\|g\|_{L_2(\Sigma_\infty)}\|u\|_{L_2(\Sigma_\infty)}, \tag{4.3.22}$$

and thus (4.3.3) holds for the wave equation on the $n$-dimensional half-space $n\geqslant 2$. □

The argument above is very transparent and shows exactly what is going on in order to gain the additional derivative on the boundary in the present case.

## *4.4 Implication on the uniform feedback stabilization of the boundary nonlinear dissipative feedback system w in (4.1.1a–c)*

We return to the feedback dissipative nonlinear system $w$ defined on the RHS of (4.1.1a–c) which we rewrite here for convenience

$$\begin{cases} w_{tt} = \mathbb{A}w & \text{in } Q; \quad (4.4.1a) \\ w(0,\,\cdot\,) = w_0,\; w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; \quad (4.4.1b) \\ w|_\Sigma = f\left[ \left.\dfrac{\partial(\mathcal{A}^{-1}w_t)}{\partial\nu}\right|_\Gamma \right] & \text{in } \Sigma. \quad (4.4.1c) \end{cases}$$

We now specialized the abstract uniform stabilization Theorem 3.1 to the present boundary dissipative feedback problem (4.4.1). To this end, we note that:

(i) the structural assumption (H.1) holds in the setting of Subsect. 4.1;

(ii) the required exact controllability assumption (H.4) of the linear open-loop $v$-problem (4.1.1a–c) (LHS) also holds on the space $Y$ in (4.1.2) within the class of $L_2(0,T;U)$-controls, $U = L_2(\Gamma)$, $T > 0$ sufficiently large, by virtue of Theorem 4.1.2;

(iii) the boundedness assumption (H.5) of the open-loop boundary $\rightarrow$ boundary map $B^*L$ is guaranteed by the (nontrivial) Theorem 4.1.4.

Thus, under assumptions (H.2) and (H.3) (Sect. 3) on the nonlinear function $f$, with $U = L_2(\Gamma)$, we obtain a nonlinear uniform stabilization result.

**Theorem 4.4.1.** *Let the function $f$ in* (4.4.1c) *satisfy assumptions* (H.2) *and* (H.3) *of Sect.* 3, *with $U = L_2(\Gamma)$. Then the conclusion of Theorem* 3.1 *applies to the nonlinear feedback $w$-problem* (4.1.1a–c) (RHS). *Thus, if $s(t)$ is the solution of the nonlinear* ODE *with $q$ explicitly constructed in terms of the data of the problem, we have*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{L_2(\Omega)\times H^{-1}(\Omega)} \leqslant s(t) \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{L_2(\Omega)\times H^{-1}(\Omega)} \searrow 0 \text{ as } t \nearrow +\infty. \tag{4.4.2}$$

*Remark 4.4.1.* The above result can be extended to the case where the equation also contains an interior dissipative term

$$w_{tt} = \mathbb{A}w + \mathbb{R}(w), \tag{4.4.3}$$

along with (4.4.1b–c). With $H^{-1}(\Omega) \equiv [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see (4.1.2), we have $(u_1, u_2)_{H^{-1}(\Omega)} = (\mathcal{A}^{-1}u_1, u_2)_{L_2(\Omega)}$. The nonlinear operator $\mathbb{R}$ is assumed to satisfy two assumptions:

(r.1)

$$\mathbb{R} : \text{continuous } L_2(\Omega) \to H^{-1}(\Omega). \tag{4.4.4}$$

(r.2) There exists a Frechet differentiable operator $\Pi : L_2(\Omega) \to$ real line, with $\Pi(\omega) \geqslant 0$, $\omega \in L_2(\Omega)$, such that

$$(\mathbb{R}(w), z)_{H^{-1}(\Omega)} = (\mathcal{A}^{-1}\mathbb{R}(\omega), z)_{L_2(\Omega)} = -(\Pi'(\omega), z)_{L_2(\Omega)}, \tag{4.4.5}$$

where $\Pi'$ is the Frechet derivative of $\Pi$. Next, re-labelling $\mathbb{R}$ as $\mathbb{R} = (\mathbb{R}\mathcal{A})\mathcal{A}^{-1} = \mathbb{R}_0\mathcal{A}^{-1}$, $R_0 \equiv \mathbb{R}\mathcal{A}$, we can rewrite (4.4.5) since $\mathcal{A}$ is (positive) self-adjoint, as

$$(\mathbb{R}(\omega), z)_{H^{-1}(\Omega)} = (\mathbb{R}_0(A^{-1}\omega), \mathcal{A}^{-1}z)_{L_2(\Omega)} = -(\Pi'(\omega), z)_{L_2(\Omega)}, \tag{4.4.6}$$

and then, if we let $\mathbb{R}_0 = (\widetilde{\mathbb{R}}_0)'$, with $\widetilde{\mathbb{R}}_0 : L_2(\Omega) \to$ real line, we can take

$$\Pi(\omega) = -\widetilde{\mathbb{R}}_0(\mathcal{A}^{-1}\omega) \text{ so that } (\Pi'(\omega), z) = -(\mathbb{R}_0(\mathcal{A}^{-1}\omega), \mathcal{A}^{-1}z)_{L_2(\Omega)}, \tag{4.4.7}$$

as required. A class of examples includes

$$(\widetilde{\mathbb{R}}_0)(s) = \int_\Omega |s|^q d\Omega = \int_\Omega |s|^{q-2}|s|^2 d\Omega, \tag{4.4.8}$$

whereby then

$$\mathbb{R}_0(s) \equiv (\widetilde{\mathbb{R}}_0)'(s) = q|s|^{q-2}s, \quad \mathbb{R}(\omega) = q|\mathcal{A}^{-1}\omega|^{q-2}\mathcal{A}^{-1}\omega. \tag{4.4.9}$$

For $n = 2, 3$, we can allow any $q \in [1, \infty)$ (on the strength of Sobolev embedding).

*Remark 4.4.2.* The nonlocal character of $\mathbb{R}$ (containing the term $\mathcal{A}^{-1}w$) is in line with the nonlocality of the feedback $\mathcal{A}^{-1}$. For the case of the Kirchhoff equation of Subsect. 5.2, the corresponding $\mathbb{R}$ will be local. □

The proof of uniform stabilization of (4.4.3), (4.4.1b–c) under the assumptions of the nonlinear term $\mathbb{R}(w)$ of the present remark will be given in [52]. It depends, among other things, on a unique continuation property of the wave equation with a time-space-dependent potential [56, 91]. □

## 4.5 Implication on exact controllability of the (linear) dissipative system under boundary control

We return to the $w$-dissipative hyperbolic problem (in the linear case $f(u) \equiv u \in L_2(\Gamma)$) on the RHS of (4.1.1a–c), which we now turn into a controlled problem under boundary control. Thus, we consider

$$\begin{cases} y_{tt} = \mathbb{A}y & \text{in } Q; & (4.5.1\text{a}) \\ y(0,\cdot) = y_0,\ y_t(0,\cdot) = y_1 & \text{in } \Omega; & (4.5.1\text{b}) \\ y|_\Sigma = \left.\dfrac{\partial(\mathcal{A}^{-1}y_t)}{\partial\nu}\right|_\Gamma + u & \text{in } \Sigma, & (4.5.1\text{c}) \end{cases}$$

with $\mathbb{A}$ defined by (1.1.0) and $\mathcal{A}$ defined by (4.1.6). In (4.5.1c), $u$ is the Dirichlet boundary control. As an immediate consequence of Theorem 4.1.4 and (1.3.8d), we have the following corollary.

**Corollary 4.5.1.** *Consider the open-loop $v$-problem on the* LHS *of* (4.1.1a–c)*. Let $T > 0$ be sufficiently large. Given any* I.C. $\{v_0, v_1\} \in L_2(\Omega) \times H^{-1}(\Omega)$, *let $g$ be the $L_2(0,T;L_2(\Gamma))$-Dirichlet control that steers $\{v_0, v_1\}$ to rest* $\{0, 0\}$ *at time $T$, i.e., $g$ is such that the corresponding solution of the $v$-problem satisfies $v(T) = v_t(T) = 0$. [This is guaranteed by Theorem* 4.1.2.] *Then, with reference to the $y$-problem* (4.5.1a–c)*, the Dirichlet boundary control*

$$u = g - \left.\frac{\partial(\mathcal{A}^{-1}v_t)}{\partial\nu}\right|_\Gamma \in L_2(0,T;L_2(\Gamma)) \tag{4.5.2}$$

*steers the initial condition* $\{y_0, y_1\} \equiv \{v_0, v_1\} \in L_2(\Omega) \times H^{-1}(\Omega)$ *to rest* $\{0, 0\}$ *at the same time $T$, i.e., $u$ is such that the corresponding solution of the dissipative controlled $y$-problem in* (4.5.1a–c) *satisfies* $y(T) = y_t(T) = 0$.

*Proof.* For $u = 0$ and $\{y_0, y_1\} \in Y$, the closed-loop boundary regularity

$$\frac{\partial(\mathcal{A}^{-1}y_t)}{\partial\nu}|_\Gamma \in L_2(0,T;L_2(\Gamma))$$

is the easy result (1.3.8d). For the open-loop $v$-problem on the LHS of (4.1.1a–c) with $g \in L_2(0,T;L_2(\Gamma))$ and $\{v_0, v_1\} = 0$, the property that $\frac{\partial(\mathcal{A}^{-1}v_t)}{\partial\nu}|_\Gamma \in L_2(0,T;L_2(\Gamma))$ is precisely statement (4.1.16) of Theorem 4.1.4. Then the $v$-problem in (4.1.1a–c) and the $y$-problem (4.5.1a–c) with $\{y_0, y_1\} = \{v_0, v_1\}$ and $u$ as in (4.5.2) coincide. □

# 5 Corollary of Section 4: The Multidimensional Kirchhoff Equation with 'Moments' Boundary Control and Normal Derivatives of the Velocity as Boundary Observation

In this section, we consider the hyperbolic Kirchhoff equation on an open bounded domain $\Omega$, dim $\Omega \geqslant 1$, with boundary control acting on the 'moment' boundary conditions. Because of the special nature of the boundary conditions, this mixed PDE problem can be converted into a wave equation problem—more precisely, the $z$-problem (5.1.11) in Subsect. 5.1—modulo lower order terms. Thus, the positive results of Subsect. 5.1 can be invoked. As a result, we likewise obtain that $B^*L \in \mathcal{L}(L_2(0,T;U))$ for the present class of Kirchhoff equations.

## *5.1 Preliminaries. The operator $B^*L$*

**Linear open-loop and nonlinear closed-loop dissipative systems.**

Let $\mathbb{A}$ be the second order differential expression in (1.1.0). In this subsection, $\Omega$ is an open bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with sufficiently smooth boundary $\Gamma$. We consider the open-loop Kirchhoff equation in $\Omega$, with boundary control acting in the 'moment' boundary condition (actually, the physical moment, in dim $\Omega \geqslant 2$, is a slight modification of our boundary condition), and its corresponding closed-loop dissipative system:

$$v_{tt} - \gamma \mathbb{A} v_{tt} + \mathbb{A}^2 v = 0; \quad w_{tt} - \gamma \mathbb{A} w_{tt} + \mathbb{A}^2 w = 0 \qquad \text{in } Q; \tag{5.1.1a}$$

$$v(0,\cdot) = v_0, v_t(0,\cdot) = v_1; \quad w(0,\cdot) = w_0, w_t(0,\cdot) = w_1 \qquad \text{in } \Omega; \tag{5.1.1b}$$

$$v|_\Sigma \equiv 0, \ \mathbb{A} v|_\Sigma = g; \quad w|_\Sigma \equiv 0, \mathbb{A} w|_\Sigma = f\Big(-\frac{\partial w_t}{\partial \nu}\big|_\Gamma\Big) \qquad \text{in } \Sigma, \tag{5.1.1c}$$

with $Q \equiv (0,T] \times \Omega$; $\Sigma \equiv (0,T] \times \Gamma$. In (5.1.1a), $\gamma$ is a positive constant, $\gamma > 0$ (this is critical to make (5.1.1) hyperbolic). By $\frac{\partial}{\partial \nu}$ we actually denote the co-normal derivative with respect to $\mathbb{A}$, as in Subsect. 4.1.

**Regularity, exact controllability of the $v$-problem; uniform stabilization of the $w$-problem for $f \equiv$ identity.**

References for this subsection include [41, 14]. We begin by introducing the (state) space of optimal regularity

$$Y \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega), \tag{5.1.2}$$

where $\mathcal{A}\psi = -\mathbb{A}\psi$ as in (4.1.6). For the stabilization result, we topologize $Y$ with an *equivalent* norm, in which case we use the notation

$$Y_\gamma \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}); \tag{5.1.3a}$$

$$(f_1, f_2)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = ((I + \gamma\mathcal{A}^{\frac{1}{2}})f_1, f_2)_{L_2(\Omega)}, f_1, f_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega). \tag{5.1.3b}$$

**Theorem 5.1.1 (**regularity [41]). *Regarding the $v$-problem* (5.1.1)*, with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds for each $T > 0$ : the map*

$$\begin{aligned} &L : g \to Lg \equiv \{v, v_t\} \quad \textit{is continuous} \\ &\quad L_2(\Sigma) \to C([0,T];\ Y \equiv [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)). \end{aligned} \tag{5.1.4}$$

(*The definition of $L$ given here is in line with the abstract definition of the operator $L$ throughout this paper.*)

**Theorem 5.1.2 (**exact controllability [41, 14]). *Given any initial condition $\{v_0, v_1\} \in Y$ and $T > 0$ sufficiently large, then there exists a $g \in L_2(\Sigma)$ such that the corresponding solution of the $v$-problem* (5.1.1) *satisfies $\{v(T), v_t(T)\} = 0$.*

**Theorem 5.1.3 (**uniform stabilization [41, 14]). *With reference to the $w$-problem* (5.1.1)*, we have*

(i)

$$\textit{the map } \{w_0, w_1\} \in Y_\gamma \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \to \{w(t), w_t(t)\} \tag{5.1.5}$$

*defines a s.c. contraction semigroup $e^{At}$ on $Y_\gamma$;*

(ii)

$$\mathbb{A}w|_\Sigma = -\frac{\partial w_t}{\partial \nu} \in L_2(0, \infty; L_2(\Gamma)) \tag{5.1.6}$$

*continuously in $\{w_0, w_1\} \in Y_\gamma$;*

(iii) *there exist constants $M \geqslant 1$ and $\delta > 0$ such that*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{Y_\gamma} = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{Y_\gamma} \leqslant Me^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{Y_\gamma}, \quad t \geqslant 0. \tag{5.1.7}$$

This result was first shown in [41] for $\Omega$ strictly convex. Then this geometrical condition was eliminated in [14]. All three theorems above are obtained by PDE hard analysis energy methods (energy multipliers). As usual, the most challenging result to prove is Theorem 5.1.3 on uniform stabilization.

**Abstract model of $v$-problem [41].**

Let $\mathcal{A}$ and $D$ be the operators in (4.1.6). Then the abstract model for the $v$-problem in (5.1.1) is [41, Equations (2.7) and (2.9), p. 70]

$$v_{tt} = -(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2[v+\mathcal{A}^{-1}Dg]; \quad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A\begin{bmatrix} v \\ v_t \end{bmatrix} + Bg; \tag{5.1.8}$$

$$\begin{aligned} A &= \begin{bmatrix} 0 & I \\ -(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 & 0 \end{bmatrix}; \\ Bg &= \begin{bmatrix} 0 \\ -(I+\gamma\mathcal{A})^{-1}\mathcal{A}Dg \end{bmatrix}; \quad B^*\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = D^*\mathcal{A}x_2. \end{aligned} \tag{5.1.9}$$

With $B^*$ defined by $(Bg_2, x)_{Y_\gamma} = (g_2, B^*x)_{L_2(\Gamma)}$ with respect to the $Y_\gamma$-topology in (5.1.3), we readily find the expression in (5.1.9).

**Reduction of $v$-model to a wave equation model, modulo lower order terms.**

**The operator $B^*L$.**

With $y_0 = \{v_0, v_1\} = 0$, we see that $B^*L : g_2 \to \frac{\partial v_t}{\partial \nu}$:

$$\begin{aligned} B^*Lg_2 = B^*\begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} &= -D^*\mathcal{A}v_t(t; y_0 = 0) \\ &= \frac{\partial v_t}{\partial \nu}(t; y_0 = 0), \end{aligned} \tag{5.1.10}$$

recalling the standard property that $D^*\mathcal{A} = -\frac{\partial}{\partial \nu}$ on $H_0^1(\Omega)$.

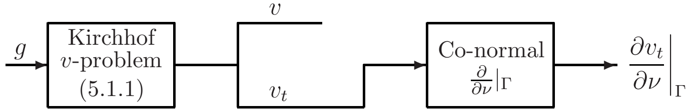


**Fig. 4** Open-loop boundary control → boundary observation $\left.\frac{\partial v_t}{\partial \nu}\right|_{\Gamma}$.

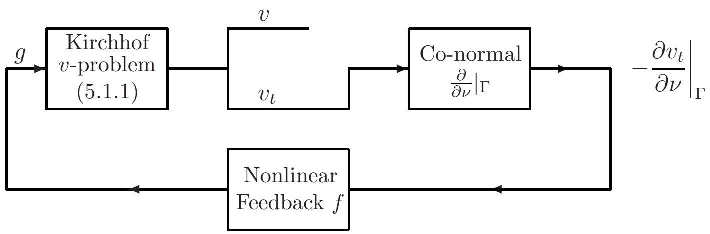


**Fig. 5** The corresponding closed-loop boundary dissipative nonlinear problem $\{w, w_t\}$.

**Goal.**

Our goal in this section is to show the following result.

**Theorem 5.1.4.** *For the v-problem* (5.1.1) *we have*

$$B^* L \in \mathcal{L}(L_2(0,T;L_2(\Gamma))). \tag{5.1.11}$$

*Proof. Reduction of v-model to a wave-model.* Using [41, (C.3), p. 100]

$$(I+\gamma\mathcal{A})^{-1}\mathcal{A}^2 = \frac{\mathcal{A}}{\gamma} - \frac{1}{\gamma^2}I + \frac{1}{\gamma^2}(I+\gamma\mathcal{A})^{-1} \text{ on } \mathcal{D}(\mathcal{A}) \tag{5.1.12}$$

in the $v$-equation (5.1.8), we find

$$v_{tt} = -\frac{\mathcal{A}v}{\gamma} - \frac{Dg}{\gamma} + \left[\frac{I}{\gamma^2} - \frac{(I+\gamma\mathcal{A})^{-1}}{\gamma^2}\right](v + \mathcal{A}^{-1}Dg), \tag{5.1.13}$$

where $v|_\Sigma \equiv 0$ by (5.1.4). Motivated by (5.1.13), we then introduce the abstract equation

$$u_{tt} = -\frac{\mathcal{A}u}{\gamma} - \frac{Dg}{\gamma}, \text{ or } \begin{cases} u_{tt} = \frac{1}{\gamma}\,\mathbb{A}u - \frac{1}{\gamma}Dg & \text{in } Q; & (5.1.14a)\\ u(0,\,\cdot\,) = 0,\ u_t(0,\,\cdot\,) = 0 & \text{in } \Omega; & (5.1.14b)\\ u|_\Sigma = 0 & \text{in } \Sigma. & (5.1.14c)\end{cases}$$

We note that the $u$-problem in (5.1.14) differs from the $v$-problem in (5.1.13) only by lower order terms in $v$, and smoother terms in $g$. Thus, the $u$-problem and the $v$-problem *possess the same regularity.* In particular, recalling (5.1.4), we have

$$\{u, u_t\} \in C([0,T];[H^2(\Omega)\cap H_0^1(\Omega)] \times H_0^1(\Omega)) \\ \text{continuously in } g \in L_2(\Sigma). \tag{5.1.15}$$

Thus, in light of (5.1.10), in order to prove (5.1.11), we equivalently establish that: with reference to the $u$-problem (5.1.14), we have

$$\text{the map } g \to \frac{\partial u_t}{\partial \nu} \text{ is continuous } L_2(\Sigma) \to L_2(\Sigma). \tag{5.1.16}$$

Indeed, statement (5.1.16) follows at once, if we introduce the new variable $z = u_t \in C([0,T]; H_0^1(\Omega))$, continuously in $g \in L_2(\Sigma)$. Then the $u$-PDE problem in (3.1.14) becomes essentially the $z$-PDE problem in (4.1.11), rewritten here for convenience:

$$z_{tt} = -\mathcal{A}z + Dg_t \begin{cases} z_{tt} = \mathbb{A}z + Dg_t & \text{in } Q; & & (5.1.17a) \\ z(0,\cdot) = 0, \;\; z_t(0,\cdot) = z_1 & \text{in } \Omega; & z = u_t & (5.1.17b) \\ z|_\Sigma \equiv 0 & \text{in } \Sigma. & & (5.1.17c) \end{cases}$$

with same *a priori* regularity as in (5.1.10): $z = u_t \in C([0,T]; H_0^1(\Omega))$. For this $z$-problem, the statement

$$\text{the map } g \to \frac{\partial z}{\partial \nu} \text{ is continuous } L_2(\Sigma) \to L_2(\Sigma), \tag{5.1.18}$$

*equivalent* to (5.1.16) has been proved in Subsect. 4.1, Theorem 4.1.4. Hence the desired conclusion (5.1.11) is established and Theorem 5.1.4 is proved. □

## *5.2 Implication on the uniform feedback stabilization of the boundary nonlinear dissipative feedback system w in (5.1.1a–c)*

In this subsection, we focus on the $w$-problem (5.1.1a–c). We seek to specialize to it the abstract uniform stabilization Theorem 3.1. To this end, we note that

(i) the structural assumption (H.1) holds in the setting of Subsect. 5.1;

(ii) the required exact controllability assumption (H.4) of the linear open-loop $v$-problem (5.1.1a–c) (LHS) also holds on the space $Y_\gamma$ in (5.1.3a) within the class of $L_2(0,T;U)$-controls with $U = L_2(\Gamma)$, $T > 0$ sufficiently large, by virtue of Theorem 5.1.2;

(iii) the boundedness assumption (H.5) of the open-loop boundary → boundary map $B^*L$ is guaranteed by the (heavy) Theorem 5.1.4.

Thus, under assumptions (H.2) and (H.3) (Sect. 3) on the nonlinear function $f$, with $U = L_2(\Gamma)$, we obtain the following nonlinear uniform stabilization result.

**Theorem 5.2.1.** *Let the function $f$ in* (5.1.1c) *satisfy assumptions* (H.2) *and* (H.3) *of Sect.* 3 *with $U = L_2(\Gamma)$. Then the conclusion of Theorem* 3.1 *applies to the nonlinear feedback $w$-problem* (5.1.1a–c) (RHS). *Thus, if $s(t)$ is the solution of the nonlinear* ODE *with $q$ explicitly constructed in terms of the data of the problem, we have*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{\mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} \leqslant s(t) \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{\mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} \searrow 0 \text{ as } t \nearrow +\infty. \quad (5.2.1)$$

*Remark 5.2.1.* The above result can be extended to the case where the equation also contains an interior dissipative term

$$w_{tt} - \gamma \mathbb{A} w_{tt} + \mathbb{A}^2 w = \mathbb{R}(w), \quad (5.2.2)$$

along with (5.1.1b–c) (see [52]). The inner product of the second component space is:

$$(u_1, u_2)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = (\mathcal{A}_\gamma u_1, u_2)_{L_2(\Omega)} \quad \text{(duality pairing)}. \quad (5.2.3)$$

The nonlinear operator $\mathbb{R}$ is assumed to satisfy two assumptions:

(r.1)

$$\mathbb{R} \text{ continuous } \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}}) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \quad (5.2.4)$$

(r.2) There exists a Frechet differentiable operator $\Pi$: $\mathcal{D}(\mathcal{A}) \to$ real line, with $\Pi(\omega) \geqslant 0$, $\omega \in \mathcal{D}(\mathcal{A})$, such that

$$(\mathcal{A}_\gamma^{-1}\mathbb{R}(\omega), z)_{\mathcal{D}(\mathcal{A}_\gamma^{\frac{1}{2}})} = (\mathbb{R}(\omega), z)_{L_2(\Omega)} = -(\Pi'(\omega), z)_{L_2(\Omega)}, \quad (5.2.5)$$

where $\Pi'$ is the Frechet derivative of $\Pi$.

Thus, in the case, we can take

$$\Pi(\omega) = -\int_\Omega \widetilde{\mathbb{R}}(\omega) d\Omega; \quad (\widetilde{\mathbb{R}})' = \mathbb{R}. \quad (5.2.6)$$

For instance,

$$\mathbb{R}(s) = q|s|^{q-2}s; \quad \widetilde{\mathbb{R}}(s) = |s|^q \geqslant 0, \quad 1 \leqslant q < \infty. \quad (5.2.7)$$

As in the case of the wave equation of Subsect. 4.4, this result depends on a unique continuation result. □

## *5.3 Implication on exact controllability of the (linear) dissipative system under boundary control*

We return to the $w$-dissipative hyperbolic problem (in the linear case $f(u) = u \in L_2(\Gamma)$) on the RHS of (5.1.1c), which we now turn into a controlled problem under boundary control. Thus, we consider

$$\begin{cases} y_{tt} - \gamma \mathbb{A} y_{tt} + \mathbb{A}^2 y = 0 & \text{in } Q; \quad (5.3.1a) \\ y(0,\cdot) = y_0, y_t(0,\cdot) = y_1 & \text{in } \Omega; \quad (5.3.1b) \\ y|_\Sigma \equiv 0, \mathbb{A} y|_\Sigma = f\left( -\left.\frac{\partial y_t}{\partial \nu}\right|_\Gamma \right) & \text{in } \Sigma, \quad (5.3.1c) \end{cases}$$

with $\mathbb{A}$ defined by (1.1.0) and $\mathcal{A}$ defined by (4.1.6), $\gamma > 0$. In (5.3.1c), $u$ is a boundary control. As an immediate consequence of Theorem 5.1.4 and (1.3.8d), we have the following corollary.

**Corollary 5.3.1.** *Consider the open-loop $v$-problem on the* LHS *of* (5.1.1a–c). *Let $T > 0$ be sufficiently large. Given any* I.C. $\{v_0, v_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, *let $g$ be the $L_2(0,T;L_2(\Gamma))$-boundary control that steers $\{v_0, v_1\}$ to rest $\{0,0\}$ at time $T$, i.e., $g$ is such that the corresponding solution of the $v$-problem satisfies $v(T) = v_t(T) = 0$ [this is guaranteed by Theorem* 5.1.2]. *Then, with reference to the $y$-problem* (5.3.1a–c), *the boundary control*

$$u = g + \left.\frac{\partial y_t}{\partial \nu}\right|_\Gamma \in L_2(0,T;L_2(\Gamma)) \tag{5.3.2}$$

*steers the initial condition $\{y_0, y_1\} = \{v_0, v_1\} \in Y \equiv [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ to rest $\{0,0\}$ at the same time $T$, i.e., $u$ is such that the corresponding solution of the dissipative controlled $y$-problem in* (5.1.1a–c) *satisfies*

$$y(T) = y_t(T) = 0.$$

*Proof.* For $u = 0$ and $\{y_0, y_1\} \in Y$, the closed-loop boundary regularity

$$\frac{\partial y_t}{\partial \nu}|_\Gamma \in L_2(0,T;L_2(\Gamma))$$

is the easy result (1.3.8d). For the open-loop $v$-problem on the LHS of (5.1.1a–c), with $g \in L_2(0,T;L_2(\Gamma))$ and $\{v_0, v_1\} = 0$, the property that $\frac{\partial v_t}{\partial \nu}|_\Gamma \in L_2(0,T;L_2(\Gamma))$ is precisely statement (5.1.16) = (5.1.11) of Theorem 5.1.4. then, the $v$-problem in (5.1.1a–c) and the $y$-problem (5.3.1a–c) with $\{y_0, y_1\} = \{v_0, v_1\}$, and $u$ as in (5.3.2) coincide. □

# 6 A First Order in Time PDE Illustration: The Schrödinger Equation under Dirichlet Boundary Control and Suitably Lifted Solution as Boundary Observation

In this section, we provide a program parallel to that of Sect. 4 or 5, this time, however, involving the first order in time Schrödinger equation with Dirichlet control. Here, the solution does not encounter the challenge of the wave equation with Dirichlet control of Sect. 4. This means that for wave equation with Dirichlet control, the complete regularity theory of the corresponding mixed problem [22] (direct estimate), as well as the corresponding exact controllability/uniform stabilization theories (reverse estimates) of the literature [29, 81, 64], are not sufficient. Consequently, an *ad hoc* nontrivial proof is needed to establish the continuity of the boundary control → boundary observation map $B^*L$, as in Sect. 4. In the present case of the Schrödinger equation, the situation is quite different. The available literature already contains the key result that the present boundary control → boundary observation map $B^*L$ is bounded [19, 32, 48], albeit explicitly in the case of constant coefficients in the principal part ($\mathbb{A} = \Delta$ in (1.1.0)). As well known since the 1986 paper [22] on wave equations—and as explicitly noted a few times in [48], the *same proof* of the constant coefficient case such as $\mathbb{A}$ in (1.1.0) only produces lower order terms which are then readily absorbed in the estimates. Moreover, unlike the case of the wave equation treated in Sect. 4, we can provide in this section an addition system-theoretic result on the corresponding transfer function $\widehat{B^*L}(\lambda)$ [here $\widehat{\ }$ denotes Laplace transform], which admits a direct, operator-theoretic, independent proof, in particular not invoking the PDE-based proof for the continuity of $B^*L$ (see Subsect. 6.4 below).

## *6.1 From the Dirichlet boundary control $u$ for the Schrödinger equation solution $y$ to the boundary observation $\frac{\partial z}{\partial \nu}|_\Gamma$, via the Poisson equation lifting $z = A^{-1}y$*

Let $\mathbb{A}$ be the differential expression defined in (1.1.0).

**Linear open-loop and nonlinear closed-loop feedback dissipative systems.**

Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$, with sufficiently smooth $C^1$-boundary $\Gamma$. We consider the following open-loop problem of the Schrödinger equation defined on $\Omega$, with Dirichlet-control $u \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$, and its corresponding boundary dissipative version

$$\begin{cases} y_t = -i\mathbb{A}y; \\ y(0,\,\cdot\,) = y_0; \\ y|_\Sigma = u \in L_2(\Sigma); \end{cases} \qquad \begin{cases} w_t = -i\mathbb{A}w & \text{in } Q; & (6.1.1a) \\ w(0,\,\cdot\,) = w_0 & \text{in } \Omega; & (6.1.1b) \\ w|_\Sigma = if\left[\left.\dfrac{\partial(A^{-1}w)}{\partial\nu}\right|_\Gamma\right] & \text{in } \Sigma, & (6.1.1c) \end{cases}$$

with $Q \equiv (0,T] \times \Omega$; $\Sigma \equiv (0,T] \times \Gamma$. Moreover, the operator $A$ is defined below in (6.1.4) as $Aw = -\mathbb{A}w$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Here, $\frac{\partial}{\partial\nu}$ denotes the co-normal derivative w.r.t. $\mathbb{A}$. The nonlinear function $f$ will be specified in Subsect. 6.2 below.

**Regularity, exact controllability of the $y$-problem; uniform stability of the $w$-problem.**

Paper [32] gives a full account of the (optimal) regularity and exact controllability of the open-loop $y$-problem in (6.1.1), as well as the uniform stabilization of the corresponding closed-loop $w$-problem. Regularity issues of interest here are also contained in [20, pp. 175-177] and [46, Chapter 10].

**Theorem 6.1.1** (regularity [32, Theorem 1.2]). *Regarding the $y$-problem* (6.1.1) *with $y_0 = 0$, for each $T > 0$ the following interior regularity holds* (*the definition of $L$ given here is in line with the abstract definition of the operator $L$ throughout this paper*) :

$$\textit{the map } L : u \to Lu = y \textit{ textiscontinuous } L_2(\Sigma) \to C([0,T];H^{-1}(\Omega)). \tag{6.1.2}$$

**Theorem 6.1.2** (exact controllability [32, Theorem 1.3], [91]). *Let $T > 0$. Given $y_0 \in H^{-1}(\Omega)$, there exists $u \in L_2(0,T;L_2(\Gamma))$ such that the corresponding solution to the $y$-problem* (6.1.1) *satisfies $y(T) = 0$.*

**Theorem 6.1.3** (uniform stabilization [32, Theorems 1.4 and 1.5], [91]). *With reference to the $w$-problem in* (6.1.1) *with $f(g) = g \in L_2(\Gamma)$ (identity), we have*

(i) *the map $w_0 \in H^{-1}(\Omega) \to w(t)$ defines a s.c. contraction semigroup on $[\mathcal{D}(A^{\frac{1}{2}})]' \equiv H^{-1}(\Omega)$;*

(ii) *$w|_\Sigma \in L_2(0,\infty;L_2(\Gamma))$ continuously for $w_0 \in H^{-1}(\Omega)$;*

(iii) *there exist constants $M \geqslant 1$ and $\delta > 0$ such that*

$$\|w(t)\| \leqslant Me^{-\delta t}\|w_0\|, \ t \geqslant 0, \tag{6.1.3}$$

*with* $\| \quad \|$ *the* $H^{-1}(\Omega)$*-norm.*

All three theorems above are obtained by PDE hard analysis energy methods (suitable energy multipliers). The most challenging result to prove is Theorem 6.1.3 on uniform stabilization: this, in addition, requires a shift of topology from $H^{-1}(\Omega)$ (the space of the final result) to $H_0^1(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable that is noted below in (6.1.8), and that is needed to establish the desired regularity of $B^*L$.

**Abstract model of $y$-problem.**

We let

$$\begin{aligned} &A\psi = -\mathbb{A}\psi, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega); \\ &\varphi \equiv Dg \Longleftrightarrow \{\mathbb{A}\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g \text{ on } \Gamma\}. \end{aligned} \tag{6.1.4}$$

Then the abstract model (in additive form) of the $y$-problem (6.2.1) is [32, Equation (1.2.2)]

$$\dot{y} = iAy - iADu = iAy + Bu, \ y(0) = y_0 \in Y \equiv [\mathcal{D}(A^{\frac{1}{2}})]' \equiv H^{-1}(\Omega); \tag{6.1.5}$$

$$B = -iAD \quad \text{ hence } B^* = iD^*, \tag{6.1.6}$$

where the $*$ for $B$ and $D$ refer actually to different topologies, as the following computation yielding $B^*$ in (6.1.6) shows. Let $u, y \in Y$. Then

$$\begin{aligned} (Bu, y)_Y &= -i(ADu, y)_{[\mathcal{D}(A^{\frac{1}{2}})]'} = -i(Du, y)_{L_2(\Omega)} \\ &= -i(u, D^*y)_{L_2(\Gamma)} = (u, B^*y)_{L_2(\Gamma)}. \end{aligned} \tag{6.1.7}$$

**A 'dissipative-like,' open-loop, boundary control → boundary observation linear problem. The operator $B^*L$.**

With reference to the $y$-problem in (6.1.1), we show that $B^*L : u \to -i\,\frac{\partial z}{\partial \nu}|_\Gamma$:

$$B^*Lu = B^*y(t; y_0 = 0) = -i\left.\frac{\partial z}{\partial \nu}\right|_\Gamma, \tag{6.1.8a}$$

$$z(t) \equiv A^{-1}y(t; y_0 = 0) \in C([0,T]; \mathcal{D}(A^{\frac{1}{2}}) \equiv H_0^1(\Omega)), \tag{6.1.8b}$$

where $z$ satisfies the following dynamics—abstract equation, and corresponding PDE-mixed problem:

$$\dot{z} = iAz - iDu \qquad \begin{cases} z_t = -i\Delta z - iDu \text{ in } Q; & \text{(6.1.9a)} \\ z(0,\cdot) = z_0 = 0 \qquad \text{in } \Omega; & \text{(6.1.9b)} \\ z|_\Sigma \equiv 0 \qquad \text{in } \Sigma. & \text{(6.1.9c)} \end{cases}$$

Indeed, to obtain (6.1.8)–(6.1.9), one uses the definitions in (6.1.8) and (6.1.6),

$$\begin{aligned} B^*Lu \equiv B^*y(t; y_0 = 0) &= iD^*AA^{-1}y(t; y_0 = 0) \\ &= iD^*Az(t) = -i\frac{\partial z}{\partial \nu}, \end{aligned} \tag{6.1.10}$$

as well as the usual property $D^*A = -\frac{\partial}{\partial \nu}$ on $\mathcal{D}(A^{\frac{1}{2}}) = H_0^1(\Omega)$ from [32, Equation (1.21)]. The abstract $z$-equation in (6.1.9) follows from the abstract $y$-equation in (6.1.5) after applying $A^{-1}$ and using the definition of $z(t)$ in (6.1.8b). Since $u(t) \in H_0^1(\Omega)$, then the abstract $z$-equation yields its PDE version in (6.1.9b).

We next provide an interpretation of the new variable $z$ via the Poisson equation. From (6.1.8b) we have

$$Az = y(t; y_0 = 0); \text{ or } \quad \begin{cases} \mathbb{A}z = -y(t; y_0 = 0) \text{ in } \Omega; & \text{(6.1.11a)} \\ z|_\Gamma = 0 \qquad \text{on } \Gamma. & \text{(6.1.11b)} \end{cases}$$

$$z(t; x_0) = -\frac{1}{2\pi}\int_\Omega G(x, t; x_0)y(t, x; y_0 = 0)dx, \tag{6.1.12}$$

$G(\quad)$ being the associated Green function on $\Omega$ (with $t$ a parameter) [11]. Thus, $z$ is the solution of the corresponding Poisson equation with zero Dirichlet boundary data and with $-y$ as a forcing term.

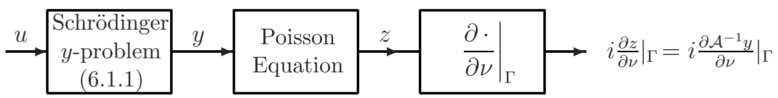


**Fig. 6** Open-loop boundary control $u \to$ boundary observation $\frac{\partial z}{\partial \nu}|_\Gamma$.

**Key boundary → boundary regularity question.**

With the optimal regularity of the variable $z$ given by (6.1.8b), we consider the corresponding Neumann trace (boundary observation) and ask the

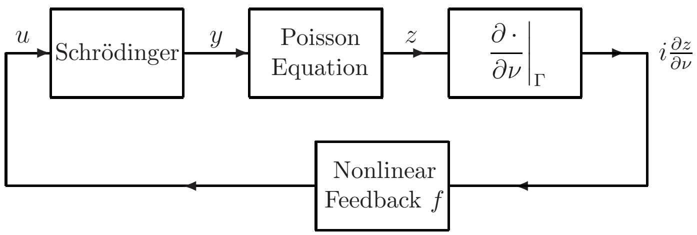


**Fig. 7** The corresponding closed-loop boundary dissipative nonlinear problem.

question (recalling (6.1.10)):

$$\begin{aligned}&\text{Does } \left.\frac{\partial z}{\partial\nu}\right|_\Gamma \in L_2(0,T;L_2(\Gamma))?\\ &\textit{i.e.},\\ &\text{Is } B^*L \text{ continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma))?\end{aligned} \tag{6.1.13}$$

The answer is affirmative. It does *not* follow directly by trace theory from the optimal interior regularity (6.1.8b) of $z$. In fact, a positive answer to question (6.1.13) would correspond to a "$\frac{1}{2}$ gain" in Sobolev-space regularity (in the space variable) over a *formal* application of trace theory to (6.1.8b).

**Theorem 6.1.4.** *With reference to* (6.1.8) *and* (6.1.13), *we have*

$$B^*L: \ \textit{continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)); \tag{6.1.14a}$$

*equivalently, with reference to* (6.1.10),

$$\textit{the map } u \to \frac{\partial z}{\partial\nu} \textit{ is continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)). \tag{6.1.14b}$$

This result (6.1.14) is explicitly stated and proved in [20, Proposition 4.2, p. 175], explicitly in the case of constant coefficients $\mathbb{A} = -\Delta$. Here, the regularity (6.1.8) for $z$ is established in [20, Equation (4.14)] by energy methods (via the multiplier $h \cdot \nabla\overline{z}, h|_\Gamma = \nu$) without first establishing the $y$-regularity (6.1.2) in Theorem 6.1.1. This result (6.1.14) also follows from [43, identity (2.1), Lemma 2.1] (built with the multiplier $h \cdot \nabla\overline{z}$) with $f = -iDu \in L_2(0,T;\mathcal{D}(A^{\frac{1}{4}-\varepsilon}))$ and the *a priori regularity* $z \in C([0,T];H_0^1(\Omega))$ in (6.1.8) for $z$: the latter uses, by contrast, the $y$-regularity (6.1.2) in Theorem 6.1.1. The two avenues chosen in [20] and [32] are very closely related and based on the same energy method and duality. The expression "double duality" was used in [20] as duality was used twice.

As noted in the introduction of Sect. 6, the case of variable coefficients (operator $\mathbb{A}$) can be proved in exactly the same way as in the case of constant coefficients, as the presence of variable coefficients only contributes lower order terms that can then be readily absorbed in the estimates. This observation is well known since the 1986 paper [22] on wave equations, and was also noted several times in [48, Remark 4.1, statement just below Equation (4.2.3), etc.].

## 6.2 Implication on the uniform feedback stabilization of the boundary nonlinear dissipative feedback system $w$ in (6.1.1a–c)

In this subsection, we focus on the $w$-problem (6.1.1a–c). We seek to specialize to it the abstract uniform stabilization Theorem 3.1. To this end, we note that

(i) the structural assumption (H.1) holds in the setting of Subsect. 6.1;

(ii) the required exact controllability assumption (H.4) of the linear open-loop $y$-problem (6.1.1a–c) (LHS) also holds on the space $Y = H^{-1}(\Omega)$ in (6.1.5) within the class of $L_2(0, T; U)$-controls, with $U = L_2(\Gamma)$, $T > 0$ arbitrary, by virtue of Theorem 6.1.2;

(iii) the boundedness assumption (H.5) of the open-loop boundary $\to$ boundary map $B^*L$ is guaranteed by Theorem 6.1.4.

Thus, under assumptions (H.2) and (H.3) (Sect. 3) on the nonlinear function $f$, with $U = L_2(\Gamma)$, we obtain the following nonlinear uniform stabilization result.

**Theorem 6.2.1.** *Let the function $f$ in* (6.1.1c) *satisfy assumptions* (H.2) *and* (H.3) *of Sect.* 3, *with $U = L_2(\Gamma)$. Then the conclusion of Theorem* 3.1 *applies to the nonlinear feedback $w$-problem* (6.1.1a–c) (RHS). *Thus, if $s(t)$ is the solution of the nonlinear* ODE *with $q$ explicitly constructed in terms of the data of the problem, we have*

$$\|w(t)\|_{H^{-1}(\Omega)} \leqslant s(t)\|w_0\|_{H^{-1}(\Omega)} \searrow 0 \text{ as } t \nearrow +\infty. \tag{6.2.1}$$

## 6.3 Implication on exact controllability of the (linear) dissipative system under boundary control

We return to the $w$-dissipative Schrödinger problem (in the linear case $f(u) \equiv u \in L_2(\Gamma)$) on the RHS of (6.1.1a–c), which we now turn into a controlled problem under boundary control. Thus, we consider

$$\begin{cases} v_t = -i\mathbb{A}v & \text{in } Q; \quad (6.3.1\text{a}) \\ v(0,\cdot) = v_0 & \text{in } \Omega; \quad (6.3.1\text{b}) \\ v|_\Sigma = i\,\dfrac{\partial(A^{-1}v)}{\partial\nu} + \mu & \text{in } \Sigma, \quad (6.3.1\text{c}) \end{cases}$$

with $\mathbb{A}$ defined by (1.1.0) and $A$ defined by (6.1.4). In (6.3.1c), $\mu$ is the Dirichlet boundary control. As an immediate consequence of Theorem 6.1.4 and (1.3.8d), we have the following corollary.

**Corollary 6.3.1.** *Consider the open-loop $y$-problem on the* LHS *of* (6.1.1a–c). *Let $T > 0$ be arbitrary. Given any* I.C. $y_0 \in H^{-1}(\Omega)$, *let $u \in L_2(0,T; L_2(\Gamma))$ be the Dirichlet control that steers $y_0$ to rest $\{0\}$ at time $T$, i.e., $u$ is such that the corresponding solution of the $y$-problem satisfies $y(T) = 0$.* [*This is guaranteed by Theorem* 4.1.2.] *Then, with reference to the $v$-problem* (6.3.1a–c), *the Dirichlet-boundary control*

$$\mu = u - i\,\frac{\partial(A^{-1}y)}{\partial\nu}\bigg|_\Gamma \in L_2(0,T; L_2(\Gamma)) \tag{6.3.2}$$

*steers the initial condition $v_0 = y_0 \in H^{-1}(\Omega)$ to rest $\{0\}$ at the same time $T$, i.e., $\mu$ is such that the corresponding solution of the dissipative controlled $v$-problem in* (6.3.1a–c) *satisfies $v(T) = 0$.*

*Proof.* (Same as the proof of Corollary 4.5.1, *mutatis mutandi.*) For $u = 0$ and $y_0 \in H^{-1}(\Omega)$, the closed-loop boundary regularity $\frac{\partial A^{-1}y}{\partial\nu}|_\Gamma \in L_2(0,T; L_2(\Gamma))$ is the (easy) result (1.3.8d). For the open-loop $y$-problem on the LHS of (6.1.1a–c), with $u \in L_2(0,T; L_2(\Gamma))$ and $y_0 = 0$, the property that $\frac{\partial(A^{-1}y)}{\partial\nu}|_\Gamma \in L_2(0,T; L_2(\Gamma))$ is precisely statement (6.1.14b) of Theorem 6.1.4. Then the $y$-problem in (6.1.1a–c) and the $v$-problem (6.3.1a–c) with $v_0 = y_0$ and $\mu$ as in (6.3.2) coincide. □

## *6.4 Asymptotic behavior of the transfer function: $(\widehat{B^*L})(\lambda) = \mathcal{O}(\lambda^{-(\frac{1}{2}-\varepsilon)})$, as positive $\lambda \nearrow +\infty$. A direct, independent proof*

In this subsection, we provide a decay rate of the transfer function $H(\lambda) \equiv \widehat{B^*L}(\lambda)$ as positive $\lambda \nearrow +\infty$. The proof is operator-theoretic and direct; in particular, it does not invoke the PDE-based result on $B^*L$ of Theorem

6.1.4. If $\widehat{\quad}$ denotes the Laplace transform, define via (1.2.2b) on $L$ and the convolution theorem

$$H(\lambda) \equiv \widehat{B^*L}(\lambda) = B^*R(\lambda, iA)B = iD^*R(\lambda; iA)(-iAD), \quad \lambda > 0 \tag{6.4.1}$$

$$= \left(D^*A^{\frac{1}{4}-\varepsilon}\right)\left[A^{\frac{1}{2}+2\varepsilon}R(\lambda, iA)\right]\left(A^{\frac{1}{4}-\varepsilon}D\right), \tag{6.4.2}$$

with $\varepsilon > 0$ arbitrary, where we have recalled (1.2.2b) for $\widehat{L}(\lambda) = R(\lambda, A)B$, and (6.1.6) for $B^*$ and $B$.

**Proposition 6.4.1.** *With reference to the transfer function $H(\lambda)$ in* (6.4.2), *the following asymptotic estimate holds with $\varepsilon > 0$ arbitrary*:

$$\|H(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \left\|\widehat{B^*L}(\lambda)\right\|_{\mathcal{L}(L_2(\Gamma))}$$
$$= \mathcal{O}\left(\frac{1}{\lambda^{\frac{1}{2}-\varepsilon}}\right), \text{ as positive } \lambda \nearrow +\infty. \tag{6.4.3}$$

*Proof. Step* 1. Recalling the basic regularity $A^{\frac{1}{4}-\varepsilon}D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$ of the Dirichlet map, we obtain from (6.4.2), where in the present proof $\|\cdot\|$ is the $\mathcal{L}(L_2(\Omega))$-norm:

$$\|H(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \left\|\widehat{B^*L}(\lambda)\right\|_{\mathcal{L}(L_2(\Gamma))}$$
$$= \mathcal{O}\left(\|A^{\frac{1}{2}+2\varepsilon}R(\lambda, iA)\|\right), \quad \lambda > 0. \tag{6.4.4}$$

*Step* 2. Since $(iA)$ is the generator of a s.c. contraction group on the space $L_2(\Omega)$, the Hille–Yosida theorem yields the resolvent bound

$$\|R(\lambda, iA)\| \leqslant \frac{1}{\lambda}, \quad \lambda > 0. \tag{6.4.5}$$

Moreover, since $(iA)R(\lambda, iA) = \lambda R(\lambda, iA) - I$, $\lambda > 0$, the above bound (6.4.5) implies:

$$\|AR(\lambda, iA)\| \leqslant \text{const}, \quad \lambda > 0. \tag{6.4.6}$$

*Step* 3. By interpolation between (6.4.5) and (6.4.6) [65], we then deduce

$$\|A^\theta R(\lambda, iA)\| \leqslant \frac{C}{\lambda^{1-\theta}}, \quad 0 \leqslant \theta \leqslant 1; \quad \|A^{\frac{1}{2}+2\varepsilon}R(\lambda, iA)\| \leqslant \frac{C}{\lambda^{\frac{1}{2}-2\varepsilon}}, \quad \lambda > 0. \tag{6.4.7}$$

Then, substituting (6.4.7) into the RHS of (6.4.4), we obtain the estimate (6.4.3) (with $2\varepsilon$ replaced by $\varepsilon$), as desired. □

This is, apparently, a sought-after result in “system theory.”

# 7 Euler–Bernoulli Plate with Clamped Boundary Controls. Neumann Boundary Control and Velocity Boundary Observation

The present section deals with the Euler–Bernoulli plate equation with “clamped” boundary controls (in any dimension), while “hinged” boundary controls will be considered in Sect. 8. In either case, the corresponding results of optimal regularity, exact controllability and uniform stabilization—all obtained by PDE energy methods—have been known since the late 80’s–early 90’s. Moreover, the regularity result $B^*L \in \mathcal{L}(L_2(0,T;U))$ is also true for each of the aforementioned Euler–Bernoulli mixed problems. This was noted in [48], explicitly in the case of constant coefficients (which, as noted several times in [48] and in the present paper, admits a direct and straightforward extension to the (space) variable coefficient case, where additional lower order terms are readily absorbed in the estimates). This result is contained in the treatments of the aforementioned literature cited as a built-in block, rather than singled out in an explicit statement. Below, we extract the necessary details from the literature, as done in [48]. After this, on the basis of Theorem 3.1, we provide a new closed-loop feedback stabilization result with nonlinear boundary feedback.

## *7.1 From the Neumann boundary control of the Euler–Bernoulli plate to the boundary observation $-\mathbb{A}z|_\Gamma$, via the Poisson lifting $z = \mathcal{A}^{-1}v_t$*

Let $\mathbb{A}$ be the differential expression defined in (1.1.0).

**Linear open-loop and nonlinear closed-loop feedback dissipative systems.**

Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ ($n = 2$, in the physical case of plates) with sufficiently smooth boundary $\Gamma$. We consider the following open-loop problem of the Euler–Bernoulli equation defined on $\Omega$, with Neumann boundary control $g_2 \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$, as well as its corresponding boundary dissipative version:

$$\begin{cases} v_{tt} + \mathbb{A}^2 v = 0; \\ v(0,\cdot) = v_0,\ v_t(0,\cdot) = v_1; \\ v|_\Sigma \equiv 0; \\ \left.\dfrac{\partial v}{\partial \nu}\right|_\Sigma = g_2; \end{cases} \qquad \begin{cases} w_{tt} + \mathbb{A}^2 w = 0 & \text{in } Q; & (7.1.1a) \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (7.1.1b) \\ w|_\Sigma \equiv 0 & \text{in } \Sigma; & (7.1.1c) \\ \dfrac{\partial w}{\partial \nu}\Big|_\Sigma = [f(\mathbb{A}(\mathcal{A}^{-1} w_t)]_\Sigma) & \text{in } \Sigma, & (7.1.1d) \end{cases}$$

with $Q = (0, T] \times \Omega$; $\Sigma = (0, T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined below in (7.1.6) as $\mathcal{A}w = \mathbb{A}^2 w$, $\mathcal{D}(\mathcal{A}) \equiv H^4(\Omega) \cap H_0^2(\Omega)$. Here, $\frac{\partial}{\partial \nu}$ denotes the co-normal derivative w.r.t. $\mathbb{A}$. The nonlinear function $f$ will be specified in Subsect. 7.2 below.

**Regularity, exact controllability of the *v*-problem; uniform stabilization of the *w*-problem.**

References for this subsection include [62, 64, 31] for the $v$-problem and [68] for the $w$-problem. These references give a full account of these three problems. We begin by introducing the (state) space (of optimal regularity)

$$X \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'; \quad [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv H^{-2}(\Omega); \quad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega). \quad (7.1.2)$$

**Theorem 7.1.1** (regularity [62, 64]). *Regarding the $v$-problem* (7.1.1)*, with $y_0 = \{v_0, v_1\} = 0$, the following regularity result holds for each $T > 0$* (*recall the definition of $L$ in* (1.2.2b))*: the map*

$$\begin{aligned} L : g_2 \to Lg_2 = \{v, v_t\} \text{ is continuous} \\ L_2(\Sigma) \to C([0,T]; X \equiv L_2(\Omega) \times H^{-2}(\Omega)). \end{aligned} \quad (7.1.3)$$

**Theorem 7.1.2** (exact controllability [63, 64, 68]). *Given any initial condition $\{v_0, v_1\} \in X$ and $T > 0$, there exists $g_2 \in L_2(\Sigma)$ such that the corresponding solution of the $v$-problem* (7.1.1) *satisfies $\{v(T), v_t(T)\} = 0$.*

**Theorem 7.1.3** (uniform stabilization [68]). *With reference to the $w$-problem* (7.1.1)*, we have*

(i) *the map $\{w_0, w_1\} \in X = L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \to \{w(t), w_t(t)\}$ defines a s.c. contraction semigroup $e^{At}$ on $X$;*

(ii)

$$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma = [\mathbb{A}(\mathcal{A}^{-1} w_t)]_\Sigma \in L_2(0, \infty; L_2(\Gamma)) \\ \text{continuously in } \{w_0, w_1\} \in X; \quad (7.1.4)$$

(iii) *there exist constants* $M \geqslant 1$ *and* $\delta > 0$ *such that*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_X = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X \leqslant M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_X, \quad t \geqslant 0. \tag{7.1.5}$$

All three theorems above are obtained by PDE hard analysis energy methods (suitable energy multipliers). As usual, the most challenging result to prove is Theorem 7.1.3 on uniform stabilization: this problem, in addition, requires a shift of topology from $X \equiv L_2(\Omega) \times H^{-2}(\Omega)$ (the space of the final result) to $H_0^2(\Omega) \times L_2(\Omega)$ (the space where the energy method works). This shift of topology is implemented by a *change of variable*: this is the same change of variable, noted below in (7.1.10), that is needed to establish the desired regularity of $B^* L$.

**Abstract model of $v$-problem.**

We let

$$\begin{aligned} &\mathcal{A}\psi = \mathbb{A}^2 \psi, \quad \mathcal{D}(\mathcal{A}) = H^4(\Omega) \cap H_0^2(\Omega); \\ &G_2 : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega), \ s \in \mathbb{R}; \end{aligned} \tag{7.1.6a}$$

$$\varphi = G_2 g_2 \Longleftrightarrow \left\{ \mathbb{A}^2 \varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = 0, \ \left.\frac{\partial \varphi}{\partial \nu}\right|_\Gamma = g_2 \right\}. \tag{7.1.6b}$$

Then the second order, respectively first order, abstract models (in additive form) of the $v$-problem (7.1.1) are [68, 31]

$$v_{tt} + \mathcal{A}v = \mathcal{A}G_2 g_2; \quad \frac{d}{dt}\begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + B g_2; \tag{7.1.7}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}; \quad B g_2 = \begin{bmatrix} 0 \\ \mathcal{A} G_2 g_2 \end{bmatrix}; \quad B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_2^* x_2, \tag{7.1.8}$$

where $*$ for $B$ and $G_2$ refer actually to different topologies. With $B^*$ defined by $(Bg_2, x)_X = (g_2, B^* x)_{L_2(\Gamma)}$ with respect to the $X$-topology, we readily find the expression in (7.1.8), since the second component of the space $X$ is $[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$.

**A 'Dissipative-Like,' Open-Loop, Boundary Control → Boundary Observation Linear Problem. The operator $B^* L$.**

With $y_0 = \{v_0, v_1\} = 0$, we show that

$$B^* L g_2 = B^* \begin{bmatrix} v(t; y_0 = 0) \\ v_t(t; y_0 = 0) \end{bmatrix} = G_2^* v_t(t; y_0 = 0) = -[\mathbb{A} z(t)]_\Gamma; \tag{7.1.9}$$

$$z(t) \equiv \mathcal{A}^{-1} v_t(t; y_0 = 0) \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega)) \quad \text{continuously in } g_2 \in L_2(\Sigma). \tag{7.1.10}$$

The new variable $z(t)$ defined in (7.1.10) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A} z = G_2 g_{2t} \qquad \begin{cases} z_{tt} + \mathbb{A}^2 z = G_2 g_{2t} & \text{in } Q; & (7.1.11a) \\ z(0, \cdot\,) = z_0 = 0; \; z_t(0, \cdot\,) = z_1 & \text{in } \Omega; & (7.1.11b) \\ z|_\Sigma \equiv 0, \left.\dfrac{\partial z}{\partial \nu}\right|_\Sigma \equiv 0 & \text{in } \Sigma. & (7.1.11c) \end{cases}$$

Indeed, to establish (7.1.9) (right), (7.1.10), one uses the definition in (7.1.9) (left), followed by (7.1.8) for $B^*$, to obtain

$$\begin{aligned} B^* L g_2 &= G_2^* v_t(t; y_0 = 0) = G_2^* \mathcal{A} \mathcal{A}^{-1} v_t(t; y_0 = 0) \\ &= G_2^* \mathcal{A} z(t) = -\mathbb{A} z(t)|_\Gamma, \end{aligned} \tag{7.1.12}$$

where, in the last step, we have recalled the usual property $G_2^* \mathcal{A} = -\mathbb{A} \cdot |_\Gamma$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega)$ [68, Equation (1.11)] and [4, Equation (1.20), p. 49]. The abstract $z$-equation is readily obtained from the abstract $v$-equation, after applying throughout $\mathcal{A}^{-1}$ and $\frac{d}{dt}$ to it, and using the definition of $z(t)$ in (7.1.10), whose *a priori* regularity in (7.1.10) follows from (7.1.3), (7.1.2). Since $z(t) \in H_0^2(\Omega)$, both boundary conditions are satisfied and the abstract $z$-equation leads to its corresponding PDE-version. By (7.1.19) below, and within the class (7.1.20), we can take $z_1 = 0$.

**Interpretation of $z$.**

We next provide an interpretation of the new variable $z$ via an elliptic problem-lifting. From (7.1.10), we have

$$\mathcal{A} z = v_t(t; y_0 = 0); \text{ or } \begin{cases} \mathbb{A}^2 = v_t(t; y_0) \text{ in } \Omega \\ z|_\Gamma = 0, \; \left.\dfrac{\partial z}{\partial \nu}\right|_\Gamma = 0. \end{cases}$$

*Remark 7.1.1.* As already noted, the change of variable $v_t \to z$ in (7.1.10) and the resulting $z$-problems in (7.1.11) are precisely the same that were used

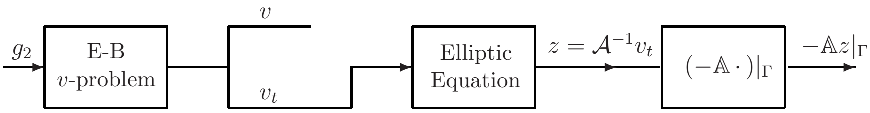


**Fig. 8** Open-loop boundary control $g_2 \to$ boundary observation $-\mathbb{A}z|_\Gamma$.

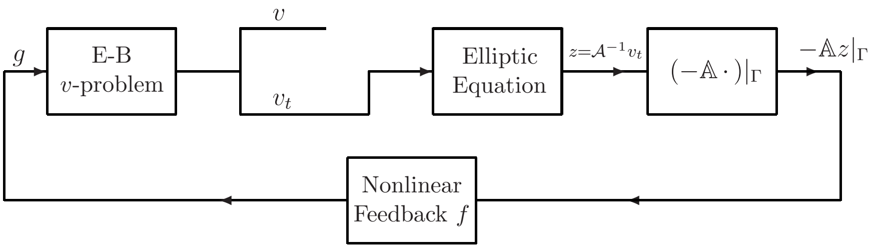


**Fig. 9** The corresponding closed-loop boundary dissipative nonlinear problem $\{w, w_t\}$.

in [68, Subsect. 2.1] in obtaining the uniform stabilization, Theorem 7.1.3, *directly*; the only difference is the specific form of the right-hand side term (thus, the letter $p$ was used in [68, Equation (2.11)], while the letter $z$ is used now for a closely related, yet not identical system). In both cases, however, a time-derivative term occurs (in our case $G_2 g_{2t}$), which will require—in [O-T.1] as well as in Step 6 in the proof of Lemma 7.1.1 below, an integration by parts in $t$, to obtain the sought-after estimate. □

**Theorem 7.1.4.** *With reference to* (7.1.9)*, we have*

$$B^* L: \text{ continuous } L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)) \tag{7.1.13a}$$

*equivalently, with reference to* (7.1.11)*,*

$$\begin{aligned} &\text{the map } g_2 \to \mathbb{A}z|_\Sigma \text{ is continuous} \\ &\qquad L_2(0,T;L_2(\Gamma)) \to L_2(0,T;L_2(\Gamma)). \end{aligned} \tag{7.1.13b}$$

As pointed out in [48], this result, though not explicitly stated, is built-in in the treatments of [68] of Theorem 7.1.3.

*Proof. Step* 1. *Basic energy identity.* As mentioned repeatedly, it suffices to confine to the constant coefficient case $\mathbb{A} = \Delta$. We return to the basic identity of the energy methods [68, Equation (2.24), p. 287], which we use with a vector field $h$ satisfying (as usual in obtaining trace regularity results [22]) the additional condition $h|_\Gamma = \nu$. Thus, with $h \cdot \nu = 1$ on $\Gamma$, for the solution $z$ of *a priori* regularity $z \in C([0,T]; H_0^2(\Omega))$ as in (7.1.10), we have

$$\frac{1}{2}\int_{\Sigma}(\Delta z)^2 d\Sigma = \mathrm{RHS}_1 + \mathrm{RHS}_2 + b_{0,T}; \tag{7.1.14}$$

$$\mathrm{RHS}_1 = \int_Q \Delta z \ \mathrm{div}[(H+H^T)\nabla z]dQ + \frac{1}{2}\int_Q z\Delta z\Delta(\mathrm{div}\ h)dQ; \tag{7.1.15}$$

$$\mathrm{RHS}_2 = -\int_Q G_2 g_{2t}\ h\cdot\nabla z\ dQ - \frac{1}{2}\int_Q G_2 g_{2t} z\ \mathrm{div}\ h\, dQ; \tag{7.1.16}$$

$$b_{0,T} = [(z_t, h\cdot\nabla z)_\Omega]_0^T + \frac{1}{2}[(z_t, z\ \mathrm{div}\ h)_\Omega]_0^T. \tag{7.1.17}$$

*Step* 2. *Estimate for* $\mathrm{RHS}_1$. From the *a priori* regularity (7.1.10) for $z$, we immediately find that

$$\mathrm{RHS}_1 = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right) \quad \forall\ g_2 \in L_2(\Sigma). \tag{7.1.18}$$

*Step* 3. *Regularity of* $z_t$. To handle $\mathrm{RHS}_2$ (by integration by parts in $t$, precisely as in the proof of the uniform stabilization Theorem 7.1.3 given in [68, p. 283-289], we need the regularity of $z_t$. By (7.1.10) and the $v$-equation (7.1.7), we obtain

$$\begin{aligned} z_t(t) &= \mathcal{A}^{-1}v_{tt} = \mathcal{A}^{-1}[-\mathcal{A}v + \mathcal{A}G_2 g_2] \\ &= -v + G_2 g_2 \in L_2(0,T;L_2(\Omega)) \text{ continuously in } g_2 \in L_2(\Sigma), \end{aligned} \tag{7.1.19}$$

by recalling that $v \in C([0,T];L_2(\Omega))$ (see (7.1.3)) and that $G_2 g_2 \in L_2(0,T;H^{\frac{3}{2}}(\Omega))$, by virtue of (7.1.6a) with $s=0$ on $G_2$ and $g_2 \in L_2(\Sigma)$.

*Step* 4. *Estimates for* $\mathrm{RHS}_2$ *and* $b_{0,T}$ *for smoother* $g_2$. Henceforth, to estimate both $\mathrm{RHS}_2$ and $b_{0,T}$, we at first take $g_2$ within the smoother class

$$g_2 \in C([0,T];L_2(\Gamma)), \quad g_2(0) = g_2(T) = 0. \tag{7.1.20}$$

This initial restriction is dictated by the fact that $z_t$ in (7.1.19) is only in $L_2$ in time.

**Lemma 7.1.1.** *In the present setting, we have*

$$RHS_2 = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right); \qquad b_{0,T} = \mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right), \tag{7.1.21}$$

*for all* $g_2$ *in the class* (7.1.20).

*Step* 5. *Proof of* (7.1.21) *for* $b_{0,T}$. First from (7.1.10) and (7.1.3), (7.1.2), we have since $v_t(0) = v_1 = 0$:

$$z(0) = 0, \quad z(T) = \mathcal{A}^{-1} v_t(T; y_0 = 0) \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^2(\Omega), \quad \text{continuously in } g_2 \in L_2(\Sigma). \tag{7.1.22}$$

Next, for $g_2$ in the class (7.1.20) used in (7.1.19), we compute since $v(0) = v_0 = 0$:

$$z_t(0) = 0, \; z_t(T) = -v(T) \in L_2(\Omega), \text{ continuously in } g_2 \in L_2(\Sigma), \tag{7.1.23}$$

where the regularity follows from (7.1.3). Using (7.1.22), (7.1.23) in (7.1.17), we readily obtain, as desired:

$$b_{0,T} = (z_t(T), h \cdot \nabla z(T))_\Omega + \frac{1}{2}(z_t(T), z(T) \text{ div } h)_\Omega = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}), \tag{7.1.24}$$

for all $g_2$ in the class (7.1.20). Thus, (7.1.21) (right) is proved.

*Step* 6. *Proof of* (7.1.21) *for* $\text{RHS}_2$. The most critical terms of $\text{RHS}_2$ to estimate is the first term in (7.1.16). As in the *direct* proof of the uniform stabilization Theorem 7.1.3 given in [68, p. 287], we integrate by parts in $t$, with $g_2$ in the class (7.1.20), thus obtaining

$$\int_Q G_2 g_{2t} \; h \cdot \nabla z \, dQ = \left[ \int_\Omega \cancel{G_2 g_2 \; h \cdot \nabla z \, d\Omega} \right]_0^T - \int_Q G_2 g_2 \; h \cdot \nabla z_t dQ, \tag{7.1.25}$$

where the first term on the right-hand side of (7.1.25) vanishes, since $g_2(0) = g_2(T) = 0$. Moreover, the usual divergence theorem [68, Equation (2.31), p. 288] yields with $h \cdot \nu = 1$:

$$\begin{aligned} &\int_0^T \int_\Omega G_2 g_2 h \cdot \nabla z_t d\Omega \, dt \\ &= \int_0^T \int_\Gamma \cancel{G_2 g_2 z_t h} \cdot \nu \, d\Gamma \, dt - \int_0^T \int_\Omega z_t h \cdot \nabla(G_2 g_2) d\Omega \, dt \\ &\quad - \int_0^T \int_\Omega G_2 g_2 z_t \text{ div } h \, d\Omega \, dt = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}), \end{aligned} \tag{7.1.26}$$

for all $g_2$ in the class (7.1.20). The indicated estimate in terms of $g_2$ in (7.1.26) follows by virtue of $z_t \in L_2(0,T; L_2(\Omega))$ (see (7.1.19)); $G_2 g_2 \in L_2(0,T; H^{\frac{3}{2}}(\Omega))$ by (7.1.6a) with $s = 0$ on $G_2$; and thus $|\nabla(G_2 g_2)| \in$

$L_2(0,T;H^{\frac{1}{2}}(\Omega))$, all bounded by the $L_2(\Sigma)$-norm of $g_2$. A similar estimate as (7.1.26) holds, *a fortiori*, for the more regular second term in the definition of $\text{RHS}_2$ in (7.1.16). Accordingly, we obtain (7.1.21) for $\text{RHS}_2$.

*Step* 7. We can then extend the estimates (7.1.21) for $\text{RHS}_2$ and $b_{0,T}$ to all $g_2 \in L_2(\Sigma)$, by density, starting from the class (7.1.20). Using these extended estimates, as well as (7.1.18) in (7.1.14), we finally obtain

$$\int_{\Sigma} (\Delta z)^2 d\Sigma = \mathcal{O}(\|g_2\|^2_{L_2(\Sigma)}) \quad \forall\, g_2 \in L_2(\Sigma), \tag{7.1.27}$$

and (7.1.13b) is proved. The proof of Theorem 4.3.4 is complete. □

## *7.2 Implication on the uniform feedback stabilization of the boundary nonlinear dissipative feedback system w in (7.1.1a–d)*

In this subsection, we focus on the $w$-problem (7.1.1a–c). We seek to specialize to it the abstract uniform stabilization Theorem 3.1. To this end, we note that

(i) the structural assumption (H.1) holds in the setting of Subsect. 7.1 (see Subsect. 1.2);

(ii) the required exact controllability assumption (H.4) of the linear open-loop $v$-problem (7.1.1a–d) (LHS) also holds on the space $X = L_2(\Omega) \times H^{-2}(\Omega)$ in (7.1.2) within the class of $L_2(0,T;U)$-controls, with $U = L_2(\Gamma)$, $T > 0$ arbitrary, by virtue of Theorem 7.1.2;

(iii) the boundedness assumption (H.5) of the open-loop boundary $\rightarrow$ boundary map $B^*L$ is guaranteed by Theorem 7.1.4.

Thus, under assumptions (H.2) and (H.3) (Sect. 3) on the nonlinear function $f$, with $U = L_2(\Gamma)$, we obtain the following nonlinear uniform stabilization result.

**Theorem 7.2.1.** *Let the function $f$ in* (7.1.1d) *satisfy assumptions (H.2) and (H.3) of Sect.* 3, *with $U = L_2(\Gamma)$. Then the conclusion of Theorem* 3.1 *applies to the nonlinear feedback $w$-problem* (7.1.1a–d) (RHS). *Thus, if $s(t)$ is the solution of the nonlinear* ODE *with $q$ explicitly constructed in terms of the data of the problem, we have*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{L_2(\Omega)\times H^{-2}(\Omega)} \leqslant \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{L_2(\Omega)\times H^{-2}(\Omega)} \searrow 0 \text{ as } t \nearrow +\infty. \quad (7.2.1)$$

## *7.3 Implication on exact controllability of the (linear) dissipative system under boundary control*

We return to the $w$-dissipative Euler–Bernoulli problem (in the linear case $f(u) = u \in L_2(\Gamma)$) on the RHS of (7.1.1a–d), which we now turn into a controlled problem under boundary control.

Thus, we consider

$$\begin{cases} y_{tt} + \mathbb{A}^2 y = 0 & \text{in } Q; & (7.3.1a) \\ y(0,\cdot) = y_0, \; y_t(0,\cdot) = y_1 & \text{in } \Omega; & (7.3.1b) \\ y|_\Sigma \equiv 0 & \text{in } \Sigma; & (7.3.1c) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma = [\mathbb{A}(\mathcal{A}^{-1}y_t)]_\Sigma) + u & \text{in } \Sigma, & (7.3.1d) \end{cases}$$

with $\mathbb{A}$ defined by (1.1.0), and $\mathcal{A}$ defined by (7.1.6). In (7.3.1d), $u$ is the boundary control. As an immediate consequence of Theorem 7.1.4 and (1.3.8d), we have the following corollary.

**Corollary 7.3.1.** *Consider the open-loop $v$-problem on the* LHS *of* (7.1.1a–d)*. Let $T > 0$ be arbitrary. Given any* I.C. $\{v_0, v_1\} \in X \equiv L_2(\Omega) \times H^{-2}(\Omega)$, *let $g_2 \in L_2(0,T;L_2(\Gamma))$ be the Neumann-boundary control that steers $\{v_0, v_1\}$ to rest $\{0,0\}$ at time $T$, i.e., $g_2$ is such that the corresponding solution of the $v$-problem satisfies $v(T) = v_t(T) = 0$ [this is guaranteed by Theorem* 7.1.2]*. Then, with reference to the $y$-problem* (7.3.1a–d)*, the boundary control*

$$u = g_2 - \mathbb{A}(\mathcal{A}^{-1}y_t)|_\Gamma \in L_2(0,T;L_2(\Gamma)), \qquad (7.3.2)$$

*steers the* I.C. $\{y_0, y_1\} = \{v_0, v_1\} \in X$ *to rest $\{0,0\}$ at the same time $T$, i.e., $u$ is such that the corresponding solution of the dissipative controlled $y$-problem in* (7.3.1a–d) *satisfies $y(T) = y_t(T) = 0$.*

*Proof.* Similar to that of Corollary 6.3.1, Corollary 5.3.1, Corollary 4.5.1, *mutatis mutandi.*

□

# 8 Euler–Bernoulli Plate with Hinged Boundary Controls. Boundary Control in the ‘Moment’ Boundary Condition and Suitably Lifted Velocity Boundary Observation

The present section deals with the Euler–Bernoulli plate equation with ‘moment’ boundary controls (in any dimension). Here also the corresponding results of optimal regularity, exact controllability, and uniform stabilization—all obtained by PDE energy methods (suitably multipliers)—have been known since the late 80’s, at least in the constant coefficient case, with further advances in exact controllability/uniform stabilization also in the variable coefficient case, where the passage from constant coefficient to variable coefficient is critical and challenging (unlike the case of optimal regularity) [96, 55, 10].

## *8.1 From the ‘moment’ boundary control of the Euler–Bernoulli plate to the boundary observation $\frac{\partial z_t}{\partial \nu}|_\Gamma$, via an elliptic lifting $z_t = \mathcal{A}^{-1} v_t$*

**Linear open-loop and nonlinear closed-loop feedback dissipative systems.**

Let, again, $\Omega$ be an open bounded domain in $\mathbb{R}^n$ ($n = 2$ in the physical case of plates) with sufficiently smooth $C^?$-boundary $\Gamma$. We consider the following open-loop problem of the Euler–Bernoulli equation defined on $\Omega$), with boundary control $g_2 \in L_2(0,T; L_2(\Gamma)) \equiv L_2(\Sigma)$, in the ‘moment’ boundary condition, as well as its corresponding boundary dissipative version:

$$\begin{cases} v_{tt} + \mathbb{A}^2 v = 0; \\ v(0,\cdot) = v_0,\ v_t(0,\cdot) = v_1; \\ v|_\Sigma \equiv 0; \\ \mathbb{A}v\Big|_\Sigma = g_2; \end{cases} \qquad \begin{cases} w_{tt} + \mathbb{A}^2 w = 0 & \text{in } Q; & (8.1.1a) \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega; & (8.1.1b) \\ w|_\Sigma \equiv 0 & \text{in } \Sigma; & (8.1.1c) \\ \mathbb{A}w\Big|_\Sigma = f\left(\frac{\partial}{\partial \nu}(\mathcal{A}^{-1} w_t)\big|_\Gamma\right) & \text{in } \Sigma, & (8.1.1d) \end{cases}$$

with $Q = (0,T] \times \Omega$; $\Sigma = (0,T] \times \Gamma$. Moreover, the operator $\mathcal{A}$ is defined below in (8.1.6) as $\mathcal{A}f = -\mathbb{A}f$; $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H^1_0(\Omega)$. Here, $\frac{\partial}{\partial \nu}$ denotes the co-normal derivative w.r.t. $\mathbb{A}$. The nonlinear function $f$ will be specified in Subsect. 8.2 below.

**Regularity, exact controllability of the $v$-problem; uniform stabilization of the $w$-problem.**

References for this subsection include [32, 35, 40, 63, 64, 59, 20]. We begin by introducing the (state) space of optimal regularity

$$Y \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \equiv H_0^1(\Omega) \times H^{-1}(\Omega). \tag{8.1.2}$$

**Theorem 8.1.1 (**regularity [32, Theorem 1.3, Equations (1.22) and (1.23), p. 203]). *Regarding the $v$-problem* (8.1.1) *with* $y_0 = \{v_0, v_1\} = 0$, *the following regularity result holds for each* $T > 0$ (*recall the definition of* $L$ *in* (1.2.2b)): *the map*

$$L : g_2 \rightarrow Lg_2 = \{v, v_t\} \text{ is continuous } L_2(\Sigma) \rightarrow C([0,T]; H_0^1(\Omega) \times H^{-1}(\Omega)); \tag{8.1.3a}$$

$$\rightarrow v_{tt} \text{ continuous } L_2(\Sigma) \rightarrow L_2(0,T; [\mathcal{D}(\mathcal{A}^{\frac{3}{2}})]' \equiv V'); \tag{8.1.3b}$$

$$V = \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) = \{h \in H^3(\Omega) : \ h|_\Gamma = \mathbb{A}h|_\Gamma = 0\} \tag{8.1.4}$$

[*warning*: *the operator* $A$ *in [32, Theorem 1.3] is* $A = \mathcal{A}^2$ *in our present notation for* $\mathcal{A}$, *see [32, Equations (1.5) and (1.6)]*].

**Theorem 8.1.2 (**exact controllability [20, 59]). *Given any initial condition* $\{v_0, v_1\} \in Y$ *and* $T > 0$, *there exists a* $g_2 \in L_2(\Sigma)$ *such that the corresponding solution of the $v$-problem* (8.1.1) *satisfies* $\{v(T), v_t(T)\} = 0$.

*Remark 8.1.1.* Exact controllability of the $v$-problem (8.1.1) with two boundary controls: $v|_\Sigma = g_1$ and $\Delta v|_\Sigma = g_2$, $g_1 \in H_0^1(0,T; L_2(\Gamma))$, $g_2 \in L_2(\Sigma)$ was previously obtained in [35, Theorem 1.2], [63, 64]. A different exact boundary controllability result with $g_1 = 0$ and $g_2 \in L_2(0,T; H^{\frac{1}{2}}(\Gamma))$, however, in the space $[H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$ was obtained in [40, Theorem 1.1]. □

**Theorem 8.1.3 (**uniform stabilization [20]). *With reference to the $w$-problem* (8.1.1), *we have*

(i) *the map* $\{w_0, w_1\} \in Y = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' \rightarrow \{w(t), w_t(t)\}$ *defines a s.c. contraction semigroup* $e^{At}$ *on* $Y$;

(ii)

$$\mathbb{A}w|_\Sigma = \frac{\partial \mathcal{A}^{-1} w_t}{\partial \nu} \in L_2(0,\infty; L_2(\Gamma)) \tag{8.1.5}$$

*continuously in* $\{w_0, w_1\} \in Y$;

(iii) *there exist constants* $M \geqslant 1$ *and* $\delta > 0$ *such that*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_Y = \left\| e^{At} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y \leqslant M e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_Y, \ t \geqslant 0. \tag{8.1.6}$$

All three theorems above are obtained by PDE hard analysis energy methods (suitable energy multipliers). As usual, the most challenging result to prove is Theorem 8.1.3 on uniform stabilization.

**Abstract model of $v$-problem.**

We let

$$\mathcal{A}\psi = -\mathbb{A}\psi, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega);$$
$$; 1mm] G_2 : H^s(\Gamma) \to H^{s+\frac{5}{2}}(\Omega), \ s \in \mathbb{R}, \tag{8.1.7}$$

$$\varphi = G_2 g_2 \Longleftrightarrow \{\mathbb{A}^2 \varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = 0, \ \mathbb{A}\varphi|_\Gamma = g_2 \text{ on } \Gamma\}, \tag{8.1.8}$$

and we recall the Dirichlet map $D : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega)$ defined in (6.1.4):

$$\varphi = D g_2 \Longleftrightarrow \{\mathbb{A}\varphi = 0 \text{ in } \Omega; \ \varphi|_\Gamma = g_2 \text{ on } \Gamma\}; \ G_2 = -\mathcal{A}^{-1} D, \tag{8.1.9}$$

where the last relationship is taken from [32, Remark 3.2, p. 211]. Then the second order, respectively first order, abstract models (in additive form) of the $v$-problem (8.1.1) are [32, 35]

$$v_{tt} + \mathcal{A}^2 v = \mathcal{A}^2 G_2 g_2 = -\mathcal{A} D g_2; \ \frac{d}{dt} \begin{bmatrix} v \\ v_t \end{bmatrix} = A \begin{bmatrix} v \\ v_t \end{bmatrix} + B g_2; \tag{8.1.10}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A}^2 & 0 \end{bmatrix}; \ B g_2 = \begin{bmatrix} 0 \\ \mathcal{A}^2 G_2 g_2 \end{bmatrix}; \ B^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_2^* \mathcal{A} x_2 = -D^* x_2, \tag{8.1.11}$$

where $*$ for $B$, and $G_2$ and $D$, refer to different topologies. With $B^*$ defined by $(Bg_2, x)_Y = (g_2, B^* x)_{L_2(\Gamma)}$ with respect to the $Y$-topology defined in (8.1.2), we readily find the expression in (8.1.11) also by virtue of $G_2 = -\mathcal{A}^{-1} D$.

**'Dissipative-like,' open-loop, boundary control → boundary observation linear problem. The operator $B^* L$.**

With $y_0 = \{v_0, v_1\} = 0$, we show that

$$B^*Lg_2 = B^*\begin{bmatrix} v(t;y_0=0) \\ v_t(t;y_0=0)\end{bmatrix} = G_2^*\mathcal{A}v_t(t;y_0=0) = -D^*v_t(t;y_0) \quad (8.1.12a)$$

$$= \frac{\partial}{\partial\nu}\mathcal{A}^{-1}v_t(t;y_0=0) = \frac{\partial}{\partial\nu}z_t(t); \quad (8.1.12b)$$

$$z(t) = \mathcal{A}^{-1}v(t;y_0=0) \in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V) \quad \text{continuously in } g_2 \in L_2(\Sigma). \quad (8.1.13)$$

Indeed, to obtain (8.1.12a–b), one uses the definition in (8.1.11) for $B^*$, followed by the usual property that $G_2^*\mathcal{A}^2 = \frac{\partial}{\partial\nu}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ [32, Lemma 3.1, Equation (3.7), p. 212] or $D^*\mathcal{A} = -\frac{\partial}{\partial\nu}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\Omega)$ [43, Equation (1.21)].

The regularity of $z(t)$ noted in (8.1.13) follows from (8.1.3a) for $v$, and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv H_0^1(\Omega)$. The new variable $z(t)$ defined in (8.1.13) satisfies the following dynamics: abstract equation, and corresponding PDE-mixed problem

$$z_{tt} + \mathcal{A}^2 z = \mathcal{A}G_2g_2$$

$$= -Dg_2 \begin{cases} z_{tt} + \Delta^2 z = \mathcal{A}G_2g_2 = -Dg_2 \text{ in } Q; & (8.1.14a) \\ z(0,\cdot\,) = 0,\ z_t(0,\cdot\,) = 0 \quad \text{in } \Omega; & (8.1.14b) \\ z|_\Sigma \equiv 0, \Delta z|_\Sigma \equiv 0 \quad \text{in } \Sigma. & (8.1.14c) \end{cases}$$

The abstract $z$-equation in (8.1.14) (left) is readily obtained from the abstract $v$-equation in (8.1.10), after applying $\mathcal{A}^{-1}$ and using the definition of $z(t)$ in (8.1.13). Since $z(t) \in \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V$ (see (8.1.4)), both boundary conditions are satisfied and the abstract $z$-equation leads to its corresponding PDE-version.

**Interpretation of $z$.**

We next provide an interpretation of the new variable $z$ via an elliptic problem-lifting. From (8.1.13), we have

$$\mathcal{A}z_t = v_t(t;y_0=0); \text{ or } \begin{cases} \mathbb{A}^2 z_t = v_t(t;y_0=0); \\ z_t|_\Gamma = \mathbb{A}z_t|_\Gamma = 0. \end{cases}$$

*Remark 8.1.2.* As already noted, the change of variable $v \to z$ in (8.1.13) and the resulting $z$-problems in (8.1.14) are precisely the same that were used in [20, Equations (2.7), (2.8), (4.3)] in obtaining there the uniform stabilization, Theorem 4.5.3, *directly*; the only difference is that, in [20, Equations (2.8), (4.3)], $g_2$ is expressed in feedback form: $g_2 = D^*\mathcal{A}p_t = \frac{\partial}{\partial\nu}p_t \in L_2(0,\infty;L_2(\Gamma))$ in the notation of [20]. Thus, the letter $p$ was used in [20],

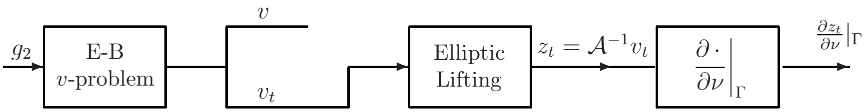


**Fig. 10** Open-loop boundary control $g_2 \to$ boundary observation $\frac{\partial z_t}{\partial \nu}\Big|_\Gamma$

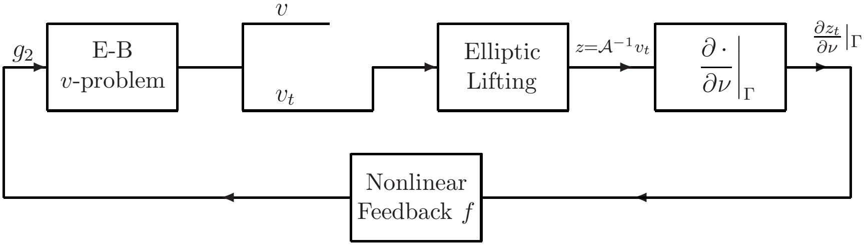


**Fig. 11** The corresponding closed-loop boundary dissipative nonlinear problem $\{w, w_t\}$

while the letter $z$ is used now. Thus, the techniques in the proof of the next, sought-after result are contained in [20] and indeed in [35, 64]. □

**Theorem 8.1.4.** *With reference to* (8.1.12)*, we have*

$$B^* L: \text{ continuous } L_2(0,T; L_2(\Gamma)) \to L_2(0,T; L_2(\Gamma)), \tag{8.1.15}$$

*equivalently, with reference to* (8.1.14)*,*

$$\begin{gathered} \text{the map } g_2 \to \frac{\partial z_t}{\partial \nu}\Big|_\Sigma \text{ is continuous} \\ L_2(0,T; L_2(\Gamma)) \to L_2(0,T; L_2(\Gamma)). \end{gathered} \tag{8.1.16}$$

We will see in the proof below that this result, though not explicitly stated, is built-in in the treatments of [20, 32, 35, 63, 64] of Theorem 8.1.1.

*Proof. Step* 1*. Basic energy identity.* As mentioned repeatedly, it suffices (for regularity purposes) to confine to the constant coefficient case $\mathbb{A} = \Delta$. We return to the basic identity of the energy method [20, 32, 35, 64], which we use with a vector field $h$ satisfying (as usual in obtaining trace regularity results [22]) the additional condition $h|_\Gamma = \nu$. Thus, with $h \cdot \nu = 1$ on $\Gamma$, for the solution $z$ of *a priori* regularity $z \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \equiv V)$ as in (8.1.13), we have (for example, [35, Equations (2.29) and (2.32)] and [32, Equations (2.1) and (2.4)])

$$\frac{1}{2}\int_{\Sigma}\left[\left(\frac{\partial\Delta z}{\partial\nu}\right)^2+\left(\frac{\partial z_t}{\partial\nu}\right)^2\right]d\Sigma=\mathrm{RHS}_1+\mathrm{RHS}_2+b_{0,T}; \tag{8.1.17}$$

$$\mathrm{RHS}_1=\int_Q H\nabla\Delta z\cdot\nabla\Delta z\,dQ+\int_Q H\nabla z_t\cdot\nabla z_t\,dQ$$

$$+\frac{1}{2}\int_Q\left(|\nabla z_t|^2-|\nabla\Delta z|^2\right)\ \mathrm{div}\ h\,dQ+\int_Q z_t\nabla(\mathrm{div}\ h)\cdot\nabla z_t dQ; \tag{8.1.18}$$

$$\mathrm{RHS}_2=-\int_Q Dg_2\nabla\Delta z\,dQ; \tag{8.1.19}$$

$$b_{0,T}=-\left[(z_t,h\cdot\nabla\Delta z)_{L_2(\Omega)}\right]_0^T. \tag{8.1.20}$$

*Step* 2. *Regularity of* $z_t$. To handle $\mathrm{RHS}_1$, we need the *a priori* regularity of $z_t$,

$$z_t=\mathcal{A}^{-1}v_t(t;y_0=0)\in C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\equiv H_0^1(\Omega))\quad\text{continuously in } g_2\in L_2(\Sigma), \tag{8.1.21}$$

as it follows from (8.1.13), (8.1.3a), and $H^{-1}(\Omega)=[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ (see (8.1.2)).

*Step* 3. *Estimate of* $\mathrm{RHS}_1$. By (8.1.13) for $z$ and (8.1.21) for $z_t$, we obtain

$$|\nabla\Delta z|,\ |\nabla z_t|\in C([0,T];L_2(\Omega)),\ \text{continuously in } g_2\in L_2(\Sigma). \tag{8.1.22}$$

Using (8.1.22) in (8.1.18) readily yields

$$\mathrm{RHS}_1=\mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right)\quad\forall\ g_2\in L_2(\Sigma). \tag{8.1.23}$$

*Step* 4. *Estimates of* $\mathrm{RHS}_2$ *and* $b_{0,T}$. From (8.1.19) and (8.1.20), by virtue of (8.1.21), (8.1.22), we readily obtain

$$\mathrm{RHS}_2+b_{0,T}=\mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right)\quad\forall\ g_2\in L_2(\Sigma). \tag{8.1.24}$$

*Step* 5. *Final estimate.* Using (8.1.23)–(8.1.24) in (8.1.17) yields

$$\frac{1}{2}\int_{\Sigma}\left[\left(\frac{\partial\Delta z}{\partial\nu}\right)^2+\left(\frac{\partial z_t}{\partial\nu}\right)^2\right]d\Sigma=\mathcal{O}\left(\|g_2\|^2_{L_2(\Sigma)}\right)\quad\forall\ g_2\in L_2(\Sigma), \tag{8.1.25}$$

and (8.1.25) *a fortiori* proves (8.1.16), as desired. The proof of Theorem 8.1.4 is complete. □

*Remark 8.1.3.* In this case, the proof of Theorem 8.1.4 is easier than the proof of uniform stabilization in [20]. But Claim 1.3.1 or Theorem 3.1 (the nonlinear version) require also exact controllability.

## 8.2 Implication on the uniform feedback stabilization of the boundary nonlinear dissipative feedback system *w* in (8.1.1a–d)

In this subsection, we focus on the $w$-problem (8.1.1a–d). We seek to specialize to it the abstract uniform stabilization Theorem 3.1. To this end, we note that

(i) the structural assumption (H.1) holds in the setting of Subsect. 8.1 (see Subsect. 1.2);

(ii) the required exact controllability assumption (H.4) of the linear open-loop $v$-problem (8.1.1a–d) (LHS) also holds on the space $Y \equiv H_0^1(\Omega) \times H^{-1}(\Omega)$ in (8.1.2) within the class of $L_2(0,T;U)$-controls with $U = L_2(\Gamma)$, $T > 0$ arbitrary, by virtue of Theorem 8.1.2;

(iii) the boundedness assumption (H.5) of the open-loop boundary $\to$ boundary map $B^*L$ is guaranteed by Theorem 8.1.4.

Thus, under assumptions (H.2) and (H.3) (Sect. 3) on the nonlinear function $f$, with $U = L_2(\Gamma)$, we obtain the following nonlinear uniform stabilization result.

**Theorem 8.2.1.** *Let the function $f$ in* (8.1.1d) *satisfy assumptions* (H.2) *and* (H.3) *of Sect.* 3, *with $U = L_2(\Gamma)$. Then the conclusion of Theorem* 3.1 *applies to the nonlinear feedback $w$-problem* (8.1.1a–d) (RHS). *Thus, if $s(t)$ is the solution of the nonlinear* ODE *with $q$ explicitly constructed in terms of the data of the problem, we have*

$$\left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|_{H_0^1(\Omega)\times H^{-1}(\Omega)} \leqslant s(t) \left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|_{H_0^1(\Omega)\times H^{-1}(\Omega)} \searrow 0 \text{ as } t \nearrow +\infty. \tag{8.2.1}$$

## 8.3 Implication on exact controllability of the (linear) dissipative system under boundary control

We return to the $w$-dissipative Euler–Bernoulli problem (in the linear case $f(u) = u \in L_2(\Gamma)$) on the RHS of (8.1.1a–d), which we turn into a controlled problem under boundary control. Thus, we consider

$$\begin{cases} y_{tt} + \mathbb{A}^2 y = 0 & \text{in } Q; & (8.3.1a) \\ y(0,\,\cdot\,) = y_0,\ y_t(0,\,\cdot\,) = y_1 & \text{in } \Omega; & (8.3.1b) \\ y|_\Sigma = 0 & \text{in } \Sigma; & (8.3.1c) \\ \mathbb{A}y|_\Sigma = \dfrac{\partial}{\partial \nu}(\mathcal{A}^{-1} y_t|_\Gamma) + u & \text{in } \Sigma, & (8.3.1d) \end{cases}$$

with $\mathbb{A}$ defined by (1.1.0) and $\mathcal{A}$ defined by (8.1.7). In (8.3.1d), $u$ is the boundary control. As an immediate consequence of Theorem 8.1.4 and (1.3.8d), we have the following corollary.

**Corollary 8.3.1.** *Consider the open-loop $v$-problem on the* LHS *of* (8.1.1a–d)*. Let $T > 0$ be arbitrary. Given any* I.C. $\{v_0, v_1\} \in Y \equiv H_0^1(\Omega) \times H^{-1}(\Omega)$, *let $g_2 \in L_2(0,T; L_2(\Gamma))$ be the 'moment' boundary control that steers $\{v_0, v_1\}$ to rest $\{0,0\}$ at time $T > 0$, i.e., $g_2$ is such that the corresponding solution of the $v$-problem satisfies $v(T) = v_t(T) = 0$ [this is guaranteed by Theorem* 8.1.2]*. Then, with reference to the $y$-problem* (8.3.1a–d)*, the boundary control*

$$u = g_2 - \frac{\partial}{\partial \nu}(\mathcal{A}^{-1} y_t) \in L_2(0,T; L_2(\Gamma)), \tag{8.3.2}$$

*steers the* I.C. $\{y_0, y_1\} = \{v_0, v_1\} \in Y \equiv H_0^1(\Omega) \times H^{-1}(\Omega)$ *to rest $\{0,0\}$ at the same time $T > 0$, i.e., $u$ is such that the corresponding solution of the dissipative controlled $y$-problem in* (8.3.1a–d) *satisfies $y(T) = y_t(T) = 0$.*

*Proof.* (See the proof of Corollaries 7.3.1, 6.3.1, 5.3.1, 4.3.1, *mutatis mutandi*) For $u \equiv 0$ and $\{y_0, y_1\} \in H_0^1(\Omega) \times H^{-1}(\Omega)$, the closed-loop boundary regularity $\frac{\partial \mathcal{A}^{-1} y_t}{\partial \nu}|_\Gamma \in L_2(0,T; L_2(\Gamma))$ is the (easy) result (1.3.8d). For the open-loop $v$-problem in the LHS of (8.1.1a–d), with $g_2 \in L_2(0,T; L_2(\Gamma))$ and $\{v_0, v_1\} = \{0,0\}$, the property that $\frac{\partial \mathcal{A}^{-1} y_t}{\partial \nu}|_\Gamma \in L_2(0,T; L_2(\Gamma))$ is precisely statement (8.1.16) of Theorem 8.1.4. Then the $v$-problem (8.1.1a–d) and the $y$-problem (8.3.1a–d) with $\{y_0, y_1\} = \{v_0, v_1\}$ and $u$ as in (8.3.2) coincide. □

## *8.4 Asymptotic behavior of the transfer function $(\widehat{B^*L})(\lambda) = \mathcal{O}(\lambda^{-(\frac{1}{2}+\varepsilon)})$, as positive $\lambda \nearrow +\infty$. A direct, independent proof*

In this subsection (as in Subsect. 6.4 for the Schrödinger equation with Dirichlet control), we provide a decay rate of the transfer function $H(\lambda) = \widehat{B^*L}(\lambda)$

as positive $\lambda \nearrow +\infty$. The proof is operator-theoretic and direct; in particular, it does not invoke the PDE-based result on $B^*L$ of Theorem 8.1.4. If $\widehat{\ }$ denotes Laplace transform, define via (1.1.2b) on $L$ and the convolution theorem:

$$H(\lambda) = \widehat{B^*L}(\lambda) = B^*(\lambda, A)B, \ \lambda > 0, \tag{8.4.1}$$

where $A$, $B$, $B^*$ are the operators in (8.1.11).

**Proposition 8.4.1.** *With reference to the transfer function* $H(\lambda)$ *in* (8.4.1), *the following asymptotic estimate holds, where* $\varepsilon > 0$ *arbitrary*:

$$\|H(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \|\widehat{B^*L}(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \mathcal{O}\left(\frac{1}{\lambda^{\frac{1}{2}-\varepsilon}}\right) \ \text{as positive } \lambda \nearrow +\infty. \tag{8.4.2}$$

*Proof. Step* 1. From $A$ in (8.1.11), we readily obtain

$$R(\lambda, A) = \begin{bmatrix} \lambda(\lambda^2 + \mathcal{A}^2)^{-1} & (\lambda^2 + \mathcal{A}^2)^{-1} \\ -\mathcal{A}^2(\lambda^2 + \mathcal{A})^{-1} & \lambda(\lambda^2 + \mathcal{A}^2)^{-1} \end{bmatrix}, \ \lambda > 0. \tag{8.4.3}$$

Combining (8.4.2) with the definition of $B$ in (8.1.11) yields

$$R(\lambda, A)B = \begin{bmatrix} -(\lambda^2 + \mathcal{A}^2)^{-1}\mathcal{A}D \\ -\lambda(\lambda^2 + \mathcal{A}^2)^{-1}\mathcal{A}D \end{bmatrix}, \ \lambda > 0. \tag{8.4.4}$$

Finally, combining (8.4.3) with the definition of $B^*$ in (8.1.11) yields

$$H(\lambda) = B^*R(\lambda, A)B = \lambda D^*\mathcal{A}(\lambda^2 + \mathcal{A}^2)^{-1}D, \quad \lambda > 0 \tag{8.4.5}$$

$$= \lambda(D^*\mathcal{A}^{\frac{1}{4}-\varepsilon})\mathcal{A}^{\frac{1}{2}+2\varepsilon}(\lambda^2 + \mathcal{A}^2)^{-1}(\mathcal{A}^{\frac{1}{4}-\varepsilon}D), \tag{8.4.6}$$

$\varepsilon > 0$, where $\mathcal{A}$ is defined in (8.1.7). [Compare with (6.4.2) for the Schrödinger equation with Dirichlet control.]

*Step* 2. Recalling the basic regularity $\mathcal{A}^{\frac{1}{4}-\varepsilon}D \in \mathcal{L}(L_2(\Gamma); L_2(\Omega))$ of the Dirichlet map, we then obtain from (8.4.6), where $\|\cdot\|$ is the $\mathcal{L}(L_2(\Omega))$-norm:

$$\|H(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \|\widehat{B^*L}(\lambda)\|_{\mathcal{L}(L_2(\Gamma))} = \lambda\mathcal{O}\left(\|\mathcal{A}^{\frac{1}{2}+2\varepsilon}(\lambda^2 + \mathcal{A}^2)^{-1}\|\right), \ \lambda > 0. \tag{8.4.7}$$

[Compare with (6.4.4).] Next, we use that: $(-\mathcal{A}^2)$ is a negative, self-adjoint operator on $L_2(\Omega)$, hence the generator of a self-adjoint contraction semigroup on $L_2(\Omega)$. Hence

$$\|R(\mu, -\mathcal{A}^2)\| = \|(\mu I + \mathcal{A}^2)^{-1}\| \leqslant \frac{1}{\mu}, \quad \mu = \lambda^2 > 0, \ \lambda > 0. \tag{8.4.8}$$

$$\|\mathcal{A}^2 R(\mu, -\mathcal{A}^2)\| \leqslant \text{const}, \tag{8.4.9}$$

where (8.4.9) follows from (8.4.8) and $\mathcal{A}^2 R(\mu, -\mathcal{A}^2) = I - \mu R(\mu, -\mathcal{A}^2)$.

*Step* 3. By interpolation between (8.4.8) and (8.4.9) [65], we then deduce

$$\|(\mathcal{A}^2)^\theta R(\mu, -\mathcal{A}^2)\| \leqslant \frac{C}{\mu^{1-\theta}}, \quad \mu = \lambda^2 > 0,\ 0 \leqslant \theta \leqslant 1. \tag{8.4.10}$$

Thus, for our case of interest $2\theta = \frac{1}{2} + 2\varepsilon$, and $\lambda > 0$, $\mu = \lambda^2$, we obtain

$$\|\mathcal{A}^{\frac{1}{2}+2\varepsilon} R(\lambda^2, -\mathcal{A}^2)\| \leqslant \frac{C}{(\lambda^2)^{1-\theta}} = \frac{C}{\lambda^{\frac{3}{2}-2\varepsilon}},\ \lambda > 0. \tag{8.4.11}$$

Substituting (8.4.11) into (8.4.7) yields (8.4.2) (with $2\varepsilon$ replaced by $\varepsilon$), as desired. □

This is, apparently, a sought-after result in ‘system theory.’

# 9 The Multidimensional Schrödinger Equation with Neumann Boundary Control on the State Space $H^1(\Omega)$ and on the State Space $L_2(\Omega)$

## *9.1 Exact controllability/uniform stabilization in $H^1(\Omega)$, $\dim \Omega \geqslant 1$*

Here, to make our point, it suffices to consider the canonical case of the multidimensional Schrödinger equation:

$$\begin{cases} iy_t - \Delta y = 0; \\ y(0,\cdot) = y_0; \\ y|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = u \in L_2(\Sigma_1); \end{cases} \qquad \begin{cases} iw_t - \Delta w = 0 \text{ in } Q; & (9.1.1a) \\ w(0,\cdot) = w_0 \quad \text{in } \Omega; & (9.1.1b) \\ w|_{\Sigma_0} \equiv 0 \qquad \text{in } \Sigma_0; & (9.1.1c) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = -w_t \text{ in } \Sigma_1, & (9.1.1d) \end{cases}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \phi$, $\Gamma_0 \neq 0$, $h \cdot \nu \leqslant 0$ in $\Gamma_0$ for a coercive smooth vector field $h(x)$ on $\Omega$. We then leave more general situations (variable coefficients in the principal part; energy level $H^1(\Omega)$-terms with variable coefficients, etc.) to the literature [87, 91], etc. We focus on the exact controllability/uniform stabilization results.

**Theorem 9.1.1** (exact controllability [67, 59, 87, 90, 91]). *Let $T > 0$ be arbitrary. Then the $y$-problem in* (9.1.1) *is exactly controllable on the state space $H^1_{\Gamma_0}(\Omega)$, with $L_2(\Sigma_1)$-controls, $\Sigma_1 = (0, T] \times \Gamma_1$.*

**Theorem 9.1.2** (uniform stabilization [67, 59, 87, 91]). (i) *The $w$-problem in* (9.1.1) *is well-posed in the semigroup sense on the space $H^1_{\Gamma_0}(\Omega)$; i.e., the map $w_0 \to w(t) = e^{A_F t} w_0$ defines a s.c. semigroup $e^{A_F t}$ on $H^1_{\Gamma_0}(\Omega)$, which is contraction in the equivalent norm of $\mathcal{D}((-A_F)^{\frac{1}{2}})$.*

(ii) *Moreover, the $w$-problem is uniformly stable on $H^1_{\Gamma_0}(\Omega)$ : there exist constants $M \geqslant 1$, $\delta > 0$ such that $\|e^{A_F t}\| \leqslant M e^{-\delta t}$, $t \geqslant 0$, in the uniform operator norm.*

*Remark 9.1.1.* First, Lebeau [59] shows the result under more general "geometric optics" conditions. Next, the case where $\cdot|_{\Sigma_0} = 0$ is replaced by $\frac{\partial \cdot}{\partial \nu}|_{\Sigma_0} = 0$ for both the $y$ and the $w$-problem is much more challenging, it requires an additional geometrical condition [57]. □

## *9.2 Exact controllability/uniform stabilization in $L_2(\Omega)$, dim $\Omega \geqslant 1$*

In this subsection, the state space will be $L_2(\Omega)$. Thus, along with the open-loop $y$-problem in (9.1.1a–d), we consider the following closed-loop boundary dissipative linear problem [58] and its corresponding nonlinear version [51]:

$$\begin{cases} iv_t + \Delta v = 0, \\ v(0,\cdot) = v_0, \\ v|_{\Sigma_0} \equiv 0; \ \dfrac{\partial v}{\partial \nu}\Big|_{\Sigma_1} = iv; \end{cases} \quad \begin{cases} iu_t + \Delta u & \text{in } Q = (0,T] \times \Omega; \\ u(0,\cdot) = u_0 & \text{in } \Omega; \\ u|_{\Sigma_0} \equiv 0; \ \dfrac{\partial u}{\partial \nu} = ig(u) & \text{in } \Sigma, \end{cases} \quad \begin{matrix} (9.2.1\text{a}) \\ (9.2.1\text{b}) \\ (9.2.1\text{c}) \end{matrix}$$

$\Sigma_i = (0, T] \times \Gamma_k$, $k = 0, 1$. These problems were introduced in [58] (linear case) and [51] (nonlinear case) and deal with (well-posedness and) uniform stabilization results on the state space $L_2(\Omega)$, a much more demanding task than the state space $H^1(\Omega)$ of Subsect. 9.1. It requires an *a priori* energy estimate at the $L_2(\Omega)$-level [58], while the natural energy space for the Schrödinger equation (where energy methods work) is $H^1(\Omega)$. The passage from $H^1(\Omega)$ to $L_2(\Omega)$ is accomplished by a pseudodifferential change of variable [58]. Subsequently, [89] provided a direct analysis of the exact controllability property of the open-loop Schrödinger equation with $L_2(0, T; L_2(\Gamma_1))$-Neumann control, for a much more general problem than the $y$-problem in (9.1.1a–c) and on a Riemannian manifold.

**Theorem 9.2.1** (exact controllability, special case of [89]). *Let $T > 0$. Assume that $\nabla d \cdot \nu \leqslant 0$ on $\Gamma_0$, for a strictly convex function $d$ (or, more generally, $h \cdot \nu \leqslant 0$ on $\Gamma_0$ for a coercive vector field $h$). Then the $y$-problem* (9.1.1a–c) *is exactly controllable on the state space $L_2(\Omega)$, within the class of $L_2(0, T; L_2(\Gamma_1))$-controls.*

**Theorem 9.2.2** (uniform stabilization [58, Sect. 11]). *With reference to the closed-loop linear dissipative $v$-problem in* (9.2.1a–c)*, we have*

(a) *the map $v_0 \to v(t)$ defines a s.c. contraction semigroup on $L_2(\Omega)$;*

(b) *assume $h \cdot \nu \leqslant 0$ on $\Gamma_0$ as in Theorem* 9.2.1; *then the strongly continuous semigroup of part* (a) *is uniformly (exponentially) stable on $L_2(\Omega)$ : there exist constant $M \geqslant 1$ and $\delta > 0$ such that*

$$\|v(t)\|_{L_2(\Omega)} \leqslant M e^{-\delta t} \|v_0\|_{L_2(\Omega)}, \quad t \geqslant 0. \tag{9.2.2}$$

Theorem 9.2.1 of the present subsection improves by one unit in the scale of Sobolev space regularity Theorem 9.1.1 of Subsect. 9.1. In the next subsections, we analyze the regularity of the operator $L$ and the regularity of the operator $B^*L$ corresponding to the open-loop $y$-problem (9.1.1a–c). A full statement of well-posedness and uniform stabilization of the nonlinear boundary feedback problem $u$ is given in [51] (following and refining the strategy of [24] for wave equations.

The regularity result is considered (at least in the negative sense for dim $\Omega \geqslant 2$) in Subsect. 9.3 below.

## *9.3 Counterexample for the multidimensional Schrödinger equation with Neumann boundary control: $L \notin \mathcal{L}(L_2(0,T; L_2(\Gamma); L_2(0,T; H^\varepsilon(\Omega)),$ $\varepsilon > 0$. A fortiori: $B^*L \notin \mathcal{L}(L_2(0,T;U))$, with $B^*$ related to the state space $H^\varepsilon(\Omega)$ and control space $U = L_2(\Gamma)$*

The present subsection complements Subsects. 9.1 and 9.2. Here, the focus will be on the multidimensional case dim $\Omega \geqslant 2$. Two main results of negative character are given, with the second being implied by the first by virtue of Theorem 2.1.

(1) With reference to the boundary $\to$ interior map $L$ defined in (1.3), we show by means of a counterexample that $L \notin \mathcal{L}(L_2(\Sigma); L_2(0, T; H^1(\Omega))$, though $H^1(\Omega)$ is the space of exact controllability/uniform stabilization, as seen in Subsect. 9.1. Even more drastically, we show that

$$L \notin \mathcal{L}(L_2(\Sigma);\ L_2(0, T; H^\varepsilon(\Omega)) \quad \forall\, \varepsilon > 0. \tag{9.3.1}$$

This negative result is the counterpart of the negative result for wave equations with $L_2(\Sigma)$-Neumann control given in [36, Counterexample, p. 294] which was already invoked in Sect. 6. The present proof is an adaptation of that given in [36].

(2) As a consequence of part (1) via Theorem 2.1, we deduce that $B^*L \notin \mathcal{L}(L_2(0,T;U))$ in the present case, where the star $*$ in $B^*$ refers to the control space $U = L_2(\Gamma)$ and the state space $H^\varepsilon(\Omega)$.

**Counterexample.**

It suffices to consider the Schrödinger equation on a 2-dimensional half-space, the setting of Subsect. 4.3, with Neumann boundary control. Hereafter, let $\Omega \equiv \mathbb{R}_2^+$ and $\Gamma = \Omega|_{x=0}$ as in (4.3.1). On $\Omega$, we consider the problem

$$
\begin{aligned}
& iv_t = v_{xx} + v_{yy} && \text{in } Q \equiv (0,\infty) \times \Omega; && (9.3.2a)\\
& v(0,\,\cdot\,) = 0 && \text{in } \Omega; && (9.3.2b)\\
& v_x|_{x=0} = g && \text{in } \Sigma \equiv (0,\infty) \times \Gamma. && (9.3.2c)
\end{aligned}
$$

*Goal*: We want to show that: given $T > 0$, there exists some $g \in L_2(0,T;L_2(\Gamma))$ such that

$$
Lg = v \notin L_2(0,T;H^\varepsilon(\Omega)) \quad \forall\, \varepsilon > 0. \tag{9.3.3}
$$

To this end, it suffices to show that there exists $g \in L_2(0,\infty;L_2(\Gamma))$ such that

$$
e^{-\gamma t}(Lg)(t) = e^{-\gamma t}v(t) \notin L_2(0,\infty;H^\varepsilon(\Omega)), \tag{9.3.4}
$$

no matter which constant $\gamma > 0$ we choose.

*Proof of* (9.3.4.). *Step* 1. Let $\widehat{v}(\tau,x,\eta)$ be the Laplace-Fourier transform of $v(t,x,y)$: Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in \mathbb{R}$, and Fourier in $y \to i\eta$, $\eta \in \mathbb{R}$, leaving $x \geqslant 0$ as a parameter. We then obtain for the solution of (9.3.2), where $\eta^2 + i\tau = (\eta^2 - \sigma) + i\gamma$:

$$
\begin{cases} i\tau\widehat{v} = \widehat{v}_{xx} - \eta^2\widehat{v} \\ \widehat{v}_x(\tau,0,\eta) = \widehat{g}(\tau,\eta) \end{cases} \qquad \text{or } \widehat{v}(\tau,x,\eta) = -\frac{\widehat{g}(\tau,\eta)}{\sqrt{(\eta^2-\sigma)+i\gamma}}\, e^{-\sqrt{(\eta^2-\sigma)+i\gamma}\;x}. \tag{9.3.5}
$$

*Step* 2. For fixed $\gamma > 0$, we define (by adaptation of [36, Equation (2.18)]) the (bad) region $\mathcal{B}^\gamma_{\sigma\eta}$ of the first quadrant of the $(\sigma,\eta)$-plane by

$$
\mathcal{B}^\gamma_{\sigma\eta} \equiv \{(\sigma,\eta) \in \mathbb{R}^2 : \sigma^2 + \eta \geqslant 1 : |\eta^2 - \sigma| \leqslant 1\}, \tag{9.3.6}
$$

comprised between the two parabolas $\eta^2 - \sigma = \pm 1$ in the first quadrant, around the parabola $\eta^2 = \sigma$. We note that in $\mathcal{B}^\gamma_{\sigma\eta}$ we have

$$\text{in } \mathcal{B}^{\gamma}_{\sigma\eta}: \ \sigma \sim \eta^2; \ |(\eta^2-\sigma)+i\gamma| \sim 1 \quad \gamma \leqslant |(\eta^2-\sigma)+i\gamma| \leqslant \sqrt{1+\gamma^2}; \tag{9.3.7a}$$

$$\sqrt{(\eta^2-\sigma)+i\gamma} = \alpha + i\beta; \quad \eta^2 - \sigma = \alpha^2 - \beta^2; \quad 2\alpha\beta = \gamma;$$
$$\alpha = \mathrm{Re}\sqrt{(\eta^2-\sigma)+i\gamma} \sim 1. \tag{9.3.7b}$$

*Step* 3. In order to establish the negative result (9.3.4), it is sufficient to prove that: there exists $g \in L_2(0,\infty; L_2(\Gamma))$ such that, recalling (9.3.5), we have

$$|\eta|^{\varepsilon}|\widehat{v}| = |\eta|^{\varepsilon} \frac{|\widehat{g}(\tau,\eta)|}{|\sqrt{(\eta^2-\sigma)+i\gamma}|} \ e^{-\mathrm{Re}\sqrt{(\eta^2-\sigma)+i\gamma}\, x} \notin L_2(0,\infty; L_2(\Omega)). \tag{9.3.8}$$

To this end, we compute as $|z^{\frac{1}{2}}| = |z|^{\frac{1}{2}}$; $z = \rho e^{i\theta}$:

$$\iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \int_0^{\infty} |\eta|^{2\varepsilon} |\widehat{v}|^2 d\mathcal{B}^{\gamma}_{\sigma\eta}$$
$$= \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \int_0^{\infty} |\eta|^{2\varepsilon} \frac{|\widehat{g}(\tau,\eta)|^2}{|(\eta^2-\sigma)+i\gamma|} \ e^{-\mathrm{Re}\sqrt{(\eta^2-\sigma)+i\gamma}\ x} dx\, d\sigma\, d\eta \tag{9.3.9}$$

$$= \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} |\eta|^{2\varepsilon} \frac{|\widehat{g}(\sigma,\eta)|^2}{|(\eta^2-\sigma)+i\gamma|} \ \frac{1}{\mathrm{Re}\sqrt{(\eta^2-\sigma)+i\gamma}} \ d\sigma\, d\eta \tag{9.3.10}$$

$$(\text{by } (9.3.7)) \sim \iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} |\eta|^{2\varepsilon} |\widehat{g}(\sigma,\eta)|^2 d\sigma\, d\eta, \tag{9.3.11}$$

where in the last step we have invoked (9.3.7a–b). Thus, it suffices to take a function $\widehat{g}(\sigma,\eta)$ which is $L_2(\mathcal{B}^{\gamma}_{\sigma\eta})$, and no better, on $\mathcal{B}^{\gamma}_{\sigma\eta}$, and zero elsewhere, to obtain for the corresponding solution $v$:

$$\iint_{\mathcal{B}^{\gamma}_{\sigma\eta}} \int_0^{\infty} |\eta|^{2\varepsilon} |\widehat{v}|^2 d\mathcal{B}^{\gamma}_{\sigma\eta} = \infty \quad \forall\ \varepsilon > 0; \tag{9.3.12}$$

hence such a $g$ is the sought-after function producing the negative conclusion (9.3.4). □

### 9.4 The operator $B^*L$, with $U = L_2(\Gamma)$ and state space $L_2(\Omega)$ of the open-loop $y$-problem (9.1.1a–d)

In this subsection, we return to the open-loop $y$-problem in (9.1.1a–d) and compute the corresponding boundary control $\to$ boundary observation operator $B^*L$ with respect to the control space $L_2(\Gamma)$ and the state space $L_2(\Omega)$. Then we establish that $B^*L$ is bounded on $L_2(0,T;L_2(\Gamma))$, at least in the (computable) case of the half-space.

**Abstract model [43, 46], [58, Subsect. 11.2]**

. The abstract model of the open-loop $y$-problem in (9.1.1a–c) is

$$iy_t = A(y - Nu); \quad y_t = -iAy - ANu \text{ on } [\mathcal{D}(A)]', \tag{9.4.1}$$

where $A$ is the positive, self-adjoint operator: $L_2(\Omega \supset \mathcal{D}(A) \to L_2(\Omega)$,

$$A\psi = -\Delta\psi,\ -\Delta\psi,\ \mathcal{D}(A) = \left\{ f \in H^2(\Omega) : f|_{\Gamma_0} = 0,\ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\}, \tag{9.4.2}$$

and $N$ is the Neumann map [46],

$$h = Ng \iff \left\{ \Delta h = 0 \text{ in } \Omega;\ h|_{\Gamma_0} = 0,\ \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma_1} = g \right\}; \tag{9.4.3}$$

$$\begin{cases} N : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega),\ s \in \mathbb{R}; & \text{(9.4.4a)} \\ N : L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\varepsilon}(\Omega) \equiv \mathcal{D}(A^{\frac{3}{4}-\varepsilon}),\ \varepsilon > 0; & \text{(9.4.4b)} \end{cases}$$

$$N^*A^*\varphi = N^*A\varphi = \begin{cases} 0 \text{ on } \Gamma_0; \\ -\varphi \text{ on } \Gamma_1, \end{cases} \varphi \in \mathcal{D}(A) \text{ [26]}; \tag{9.4.5}$$

$$B = -AN; \quad B^*g = -N^*Ag = g, \quad (Bu, g)_{L_2(\Omega)} = (u, B^*g)_{L_2(\Gamma)}, \tag{9.4.6}$$

so that the star $*$ of the adjoint refers now to the control space $L_2(\Gamma)$ and the state space $L_2(\Omega)$. Accordingly, with reference to the $y$-problem with $y_0 = 0$ we have by (9.4.6)

$$Lu = y, \quad B^*Lu = -N^*Ay|_{\Gamma_1} = y|_{\Gamma_1}. \tag{9.4.7}$$

*Is the operator $B^*L$ bounded on $L_2(0,T;L_2(\Gamma))$?* The answer is in the affirmative (unlike the case of Subsect. 9.3 on the state space $H^\varepsilon(\Omega)$), at least for the Schrödinger problem defined on the half-space.

**The half-space case.**

We return to the problem (9.3.2a–c) with control now called $g \in L_2(0,T; L_2(\Gamma)$ which we extend by zero for $t > T$. In Laplace (in time)-Fourier (in the tangential variable) the solution is given by (9.3.5)). Thus, we have (with $\Gamma_1 = \Gamma$):

$$\widehat{B^* L}g|_\Gamma = \widehat{v}(\tau, x = 0, \eta) = -\frac{\widehat{g}(\tau,\eta)}{\sqrt{(\eta^2 - \sigma) + i\gamma}}. \tag{9.4.8}$$

Since $\sqrt{(\eta^2 - \sigma) + i\gamma} \geqslant \gamma > 0$, we readily obtain

$$\iint_{\substack{\text{first}\\ \text{quadrant}}} |\widehat{v}(\tau, x = 0, \eta)|^2 d\sigma\, d\eta = \iint_{\substack{\text{first}\\ \text{quadrant}}} \frac{|\widehat{g}(\tau,\eta)|^2}{\sqrt{(\eta^2 - \sigma) + i\gamma}|^2}\, d\sigma\ d\eta \tag{9.4.9}$$

$$\leqslant \frac{1}{\gamma} \iint_{\substack{\text{first}\\ \text{quadrant}}} |\widehat{g}(\tau,\eta)|^2 d\sigma\, d\eta < \infty, \tag{9.4.10}$$

and then

$$e^{\gamma t} B^* L g = v|_\Gamma \in L_2(0,\infty; L_2(\Gamma)),\ \gamma > 0,\ g \in L_2(0,\infty; L_2(\Gamma)), \tag{9.4.11}$$

and hence

$$B^* L g = v|_\Gamma \in L_2(0,\infty; L_2(\Gamma)),\ \text{for } g \in L_2(0,\infty; L_2(\Gamma)), \tag{9.4.12}$$

as desired. It is likely that the boundedness of $B^* L$ in the present case holds for any bounded domain, but this needs to be established.

**Acknowledgment.** Research partially supported by the National Science Foundation (grant no. NSF-DMS-0104305).

## References


1. Ammari, K.: Dirichlet boundary stabilization of the wave equation. Asymptotic Anal. **30**, 117–130 (2002)
2. Balakrishnan, A.V.: Applied Functional Analysis. Springer-Verlag (1981)
3. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. SIAM J. Control Optim. **30**, 1024-1065 (1992)
4. Bartolomeo, J., Triggiani, R.: Uniform energy decay rates for Euler–Bernoulli equations with feedback operators in the Dirichlet/Neumann boundary conditions. SIAM J. Math. Anal. **22**, 46-71 (1991)
5. Bensoussan, A., Da Prato, G., Delfour, M., Mitter, S.: Representation and Control of Infinite Dimensional Systems. Vols. 1, 2. Birkhäuser (1992/93)

6. Doetsch, G.: Introduction to the Theory and Application of the Laplace Transformation. Springer-Verlag, New York–Heidelberg (1970)
7. Eller, M., Lasiecka, I., Triggiani, R.: Simultaneous exact/approximate boundary controllability of thermo-elastic plates with variable transmission coefficient. In: Shape Optimization and Optimal Design (Cambridge, 1999), pp. 109–230. Lect. Notes Pure Appl. Math. **216**, Marcel Dekker, New York (2001)
8. Flandoli, F.: Lasiecka, I., Triggiani, R.: Algebraic Riccati equations with nonsmoothing observation arising in hyperbolic and Euler–Bernoulli boundary control problems. Ann. Mat.Pura Appl. **153**, 307-382 (1988)
9. Guo, G.Zh., Luo, Y.H.: Controllability and stability of a second order hyperbolic system with collocated sensor/actuator, Syst. Control Lett. **46**, no. 1, 45-65 (2002)
10. Gulliver, R., Lasiecka, I., Littman, W., Triggiani, R.: The case for differential geometry for the control of single and coupled PDEs: The structural acoustic chamber. In: Geometric Methods in Inverse Problems and PDE Control, pp. 73–181, Springer, New York (2004)
11. Haberman, R.: Elementary Applied Partial Differential Equations, with Fourier Series and Boundary Value Problems. Prentice-Hall (1983)
12. Ho, F.L.: Boundary observability of the wave equation. C. R. Acad. Sci., Paris, Ser. I **302**, 443-446 (1986)
13. Hörmander, L.: The Analysis of Linear Partial Differential Operators. III. Springer-Verlag (1985)
14. Horn, M.A., Lasiecka, I.: Asymptotic behavior with respect to thickness of boundary stabilizing feedback for the Kirchhoff plate. J. Differ. Equations **114**, no. 2, 396-433 (1994)
15. Kreiss, H.O.: Initial boundary value problems for hyperbolic systems. Commun. Pure Appl. Math. **23**, 277-298 (1970)
16. Lagnese, J.: Boundary Stabilization of Thin Plates. SIAM Studies Appl. Math. **10**. SIAM, Philadelphia, PA (1989)
17. Lagnese, J., Lions, J.L.: Modelling, Analysis and Control of Thin Plates. Masson, Paris (1988)
18. Lieb, E.H., Loss, M.: Analysis. Am. Math. Soc., Providence (1996)
19. Lasiecka, I.: Stabilization of wave and plate-like equations with nonlinear dissipation. J. Differ. Equations **79**, no. 2, 340-381 (1989)
20. Lasiecka, I.: Exponential decay rates for the solutions of Euler–Bernoulli equations with boundary dissipation occurring in the moment only. J. Differ. Equations **95**, no. 1, 169-182 (1992)
21. Lasiecka, I.: Mathematical Control Theory of Coupled PDEs CBMS-NSF Regional Conf. Series Appl. Math. **75**. SIAM, Philadelphia, PA (2002)
22. Lasiecka, I., Lions, J.L., Triggiani, R.: Non-homogeneous boundary value problems for second order hyperbolic operators. J. Math. Pures Appl. (9) **65**, no. 2, 149-192 (1986)
23. Lasiecka, I., Lukes, D., Pandolfi, L.: Input dynamics and nonstandard Riccati equations with applications to boundary control of damped wave and plate equations. J. Optim. Theory Appl. **84**, no. 3, 549–574 (1995)
24. Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differ. Integral Equ. **6**, 507-533 (1993)
25. Lasiecka, I., Toundykov, D.: Energy decay rae for semilinear wave equations with nonlinear localized damping adn source terms. Nonlinear Anal. **64**, 1757-1797 (2006)
26. Lasiecka, I., Triggiani, R.: A cosine operator approach to modeling $L_2(0,T;L_2(\Gamma))$-boundary input hyperbolic equations, Appl. Math. Optim. **7**, no. 1, 35-93 (1981)
27. Lasiecka, I., Triggiani, R.: Regularity of hyperbolic equations under $L_2(0,T;L_2(\Gamma))$-Dirichlet boundary terms. Appl. Math. Optim. **10**, no. 3, 275-286 (1983)

28. Lasiecka, I., Triggiani, R.: Riccati equations for hyperbolic partial differential equations with $L_2(\Sigma)$-Dirichlet boundary terms. SIAM J. Control Optim. **24**, no. 5, 884-926 (1986)
29. Lasiecka, I., Triggiani, R.: Uniform exponential energy decay of wave equations in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions. J. Differ. Equations **66**, 340-390 (1987)
30. Lasiecka, I., Triggiani, R.: A lifting theorem for the time regularity of solutions to abstract equations with unbounded operators and applications to hyperbolic equations. Proc. Am. Math. Soc. **103**, 745-755 (1988)
31. Lasiecka, I., Triggiani, R.: Exact controllability of the Euler–Bernoulli equation with controls in the Dirichlet and Neumann Boundary Conditions: A nonconservative case. SIAM J. Control Optim. **27**, no. 2, 330-373 (1989)
32. Lasiecka, I., Triggiani, R.: Regularity theory for a class of nonhomogeneous Euler–Bernoulli equations: A cosine operator approach. Boll. Un. Mat. Ital. B (7) **3**, no. 1, 199–228 (1989)
33. Lasiecka, I., Triggiani, R.: Exact boundary controllability of the wave equation with Neumann boundary control. Appl. Math. Optim. **19**, 243-290 (1989)
34. Lasiecka, I., Triggiani, R.: Sharp regularity theory for second order hyperbolic equations of Neumann type. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **83**, 109-113 (1989)
35. Lasiecka, I., Triggiani, R.: Exact controllability of the Euler–Bernoulli equation with boundary controls for displacement and moment. J. Math. Anal. Appl. **146**, 1-33 (1990)
36. Lasiecka, I., Triggiani, R.: Sharp regularity theory for second order hyperbolic equations of Neumann type. I: $L_2$-nonhomogeneous data. Ann. Mat. Pura Appl. (4) **157**, 285-367 (1990)
37. Lasiecka, I., Triggiani, R.: Exact controllability of semi-linear abstract systems with application to wave and plate problems. Appl. Math. Optim. **23**, 109-154 (1991)
38. Lasiecka, I., Triggiani, R.: Regularity theory of hyperbolic equations with nonhomogeneous Neumann boundary conditions. II: General boundary data. J. Differ. Equations **94**, no. 1, 112-164 (1991)
39. Lasiecka, I., Triggiani, R.: Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory. Lect. Notes Control Informat. Sci. **164**. Springer-Verlag, Berlin (1991)
40. Lasiecka, I., Triggiani, R.: Exact controllability and uniform stabilization of Euler–Bernoulli equations with boundary control only in $\Delta w|_\Sigma$. Boll. Un. Mat. Ital. B (7) **5**, no. 3, 665-702 (1991)
41. Lasiecka, I., Triggiani, R.: Exact controllability and uniform stabilization of Kirchhoff plates with boundary control only in $\Delta w|_\Sigma$ and homogeneous boundary displacement. J. Differ. Equations **93**, no.1, 62-101 (1991)
42. Lasiecka, I., Triggiani, R.: Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions. Appl. Math. Optim. **25**, no. 2, 189-224 (1992)
43. Lasiecka, I., Triggiani, R.: Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control. Differ. Integral Equ. **5**, 521-535 (1992)
44. Lasiecka, I., Triggiani, R.: Sharp trace estimates of solutions to Kirchhoff and Euler–Bernoulli equations. Appl. Math. Optim. **28**, 277-306 (1993)
45. Lasiecka, I., Triggiani, R.: Recent advances in regularity theory of second order hyperbolic mixed problems and applications. In: Jones, Ch. (ed.) et al., Dynamics Reported. Expositions in Dynamical Systems, pp. 104-162. Springer-Verlag, Berlin (1994)

46. Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations. I, II. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambrdige (2000)
47. Lasiecka, I., Triggiani, R.: Uniform stabilization of a shallow shell model with nonlinear boundary feedbacks. J. Math. Anal. Appl. **269**, 642-688 (2002)
48. Lasiecka, I., Triggiani, R.: $L_2(\Sigma)$-regularity of the boundary to boundary operator for hyperbolic and Petrowski type PDEs. Abstr. Appl. Anal. no. 19, 1061-1139 (2003)
49. Lasiecka, I., Triggiani, R.: The operator $B^*L$ for the wave equation with Dirichlet control. Abstr. Appl. Anal. no. 7, 625-634 (2004)
50. Lasiecka, I., Triggiani, R.: Global exact controllability of semilinear wave equations by a double compactness/uniqueness argument. Discrete Contin. Dyn. Syst. Suppl. 556-565 (2005)
51. Lasiecka, I., Triggiani, R.: Well-posedness and uniform decay rates at the $L_2(\Omega)$-level of Schrödinger equations with nonlinear boundary dissipation. J. Evol. Equ. **6**, no. 3, 485-537 (2006)
52. Lasiecka, I., Triggiani, R.: Uniform energy decay rates of hyperbolic equations with nonlinear boundary and interior dissipation. Control Cybernet. (2008) [To appear]
53. Lasiecka, I., Triggiani, R., Yao, P.F.: An observability estimate in $L_2(\Omega) \times H^{-1}(\Omega)$ for second order hyperbolic equations with variable coefficients. In: Chen, Sh. (ed.) et al., Control of Distributed Parameter and Stochastic Systems. Proc. Intern. Conf. (China, 1998), pp. 71-78. Kluwer Academic Publishers, Boston, MA (1999)
54. Lasiecka, I., Triggiani, R., Yao, P.F.: Inverse/observability estimates for second order hyperbolic equations with variable coefficients. J. Math. Anal. Appl. **235**, 13-57 (1999)
55. Lasiecka, I., Triggiani, R., Yao, P.F.: Carleman estimates for a plate equation on a Riemannian manifold with energy level terms. In: Begehr, H. (ed.) et al., Analysis and Applications–ISAAC 2001. Proc. the 3rd Intern. Congress (Germany, 2001), pp. 199-236. Kluwer Academic Publishers, Dordrecht (2003)
56. Lasiecka, I., Triggiani, R., Zhang, X.: Nonconservative wave equations with unobserved Neumann B.C.: Global uniqueness and observability in one shot. Contemp. Math. **268**, 227-325 (2000)
57. Lasiecka, I., Triggiani, R., Zhang, X.: Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. I: $H^1(\Omega)$-estimates. J. Inverse Ill-Posed Probl. **12**, no. 1, 1-81 (2004)
58. Lasiecka, I., Triggiani, R., Zhang, X.: Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. II: $L_2(\Omega)$-estimates. J. Inverse Ill-Posed Probl. **12**, no. 2, 183-231 (2004).
59. Lebeau, G.: Control of the Schrödinger equation (in French). J. Math. Pures Appl. (9) **71**, no. 3, 267-291 (1992)
60. Lions, J.L.: Optimal Control of Systems Governed by Partial Differential Equations, Springer Verlag, 1971.
61. Lions, J.L.: Controle de systeme distribues singuliers. Gauthier-Vilars, Paris (1983)
62. Lions, J.L.: Un resultat de reglarite pour l'operator $(\partial^2/\partial t^2) + \Delta^2$. In: Ohya, Y. (ed.) et al., Current Topics in Partial Differential Equations. Kinokuniya, Tokyo (1986)
63. Lions, J.L.: Exact controllability, stabilization and perturbations. SIAM Rev. **30**, 1-68 (1988)
64. Lions, J.L.: Contrôllabilité exacte des systèmes distribués. Masson, Paris (1988)
65. Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications. Vols.I–III. Springer-Verlag, Berlin–Heidelberg–New York (1972/73)
66. Lukes, D.L.: Stabilizability and optimal control. Funkc. Ekvacioj, Ser. Int. **11**, 39-50 (1968)
67. Machtyngier, E.: Exact boundary controllability and stabilizability for the Schrödinger equation (in French) C. R. Acad. Sci., Paris, Ser. I **310**, no.12, 801-806 (1990)

68. Ourada, N., Triggiani, R.: Uniform stabilization of the Euler–Bernoulli equation with feedback operator only in the Neumann boundary condition. Differ. Integral Equ. **4**, 277-292 (1991)
69. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag (1983)
70. Ralston, J.: A note on a paper of Kreiss. Commun. Pure Appl. Math. **24**, 759-762 (1971)
71. Rauch, J.: $L_2$ is a continuable initial condition for Kreiss' mixed problems. Commun. Pure Appl. Math. **25**, 265-285 (1972)
72. Russell, D.: Exact boundary controllability theorems for wave and heat processes in star complemented regions. In: Roxin, E.O. (ed.) et al., Differential Games and Control Theory. Lect. Notes Pure Appl. Math. Marcel Dekker, New York **10**, 291-319 (1974)
73. Russell, D.: Controllability and stabilizability theory for linear PDEs: Recent progress and open questions. SIAM Rev. **20**, 639-739 (1978)
74. Slemrod, M.: A note on complete controllability and stabilization for linear control systems in Hilbert space. SIAM J.Control **12**, 500-508 (1974)
75. Tataru, D.: Boundary controllability for conservative PDEs. Appl. Math. Optim. **31**, 257-295 (1995)
76. Tataru, D.: On the regularity of boundary traces of wave equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 26, no.1, 185-206 (1998)
77. Taylor, A., Lay, D.: Introduction to Functional Analysis. John Wiley (1980)
78. Triggiani, R.: On the lack of exact controllability for mild solutions in Banach space. J. Math. Anal. Appl. **50**, 438-446 (1975)
79. Triggiani, R.: A note on the lack of exact controllability for mild solutions in Banach space. SIAM J. Control Optim. **15**, no. 3, 407-411 (1977)
80. Triggiani, R.: A cosine operator approach to modeling $L_2(0,T;L_2(\Gamma))$-boundary input problems for hyperbolic systems. Optim. Techn., Proc. IFIP Conf., Würzburg 1977, Part 1, Lect. Notes Control Inf. Sci. **6**, 380-390 (1978)
81. Triggiani, R.: Exact boundary controllability on $L_2(\Omega)\times H^{-1}(\Omega)$ of the wave equation with Dirichlet boundary control. Appl. Math. Optim. **18**, 241-277 (1988)
82. Triggiani, R.: Lack of uniform stabilization for non-contractive semigroups under compact perturbation. Proc. Am. Math. Soc. **105**, 375-383 (1989)
83. Triggiani, R.: Wave equation on a bounded domain with boundary dissipation: An operator approach. J. Math. Anal. Appl. **137**, 438-461 (1989)
84. Triggiani, R.: Finite rank, relatively bounded perturbations of semigroup generators. III: A sharp result on the lack of uniform stabilization. Differ. Integral Equ. **3**, no. 3, 503-522 (1990)
85. Triggiani, R.: Lack of exact controllability for wave and plate equations with finitely many boundary controls. Differ. Integral Equ. **4**, 683-705 (1991)
86. Triggiani, R.: Optimal boundary control and new Riccati equations for highly damped second order equations. Differ. Integral Equ. **7**, 1109-1144 (1994)
87. Triggiani, R.: Carleman estimates and exact boundary controllability for a system of coupled nonconservative Schrödinger equations. Rend. Ist. Mat. Univ. Trieste no.28, Suppl., 453-504 (1996)
88. Triggiani, R.: Global exact controllability on $H^1_{\Gamma_0}(\Omega)\times L_2(\Omega)$ of semilinear wave equations with Neumann $L_2(0,T;L_2(\Gamma_1))$-boundary control. In: Imanuvilov, O. (ed.) et al., Control Theory of Partial Differential Equations, pp. 273-336. Chapman & Hall (2005)
89. Triggiani, R.: Exact controllability in $L_2(\Omega)$ of the Schrödinger equation in a Riemannian manifold with $L_2(\Sigma_1)$-Neumann boundary control. In: Amann, H. (ed.) et al., Functional Analysis and Evolution Equations, pp. 613-636. Birkhäuser (2007)

90. Triggiani, R., Xu, X.: Pointwise Carleman estimates, global uniquess, observability and stabilization for nonconservative Schrödinger equations on Riemannian manifolds at the $H^1(\Omega)$-level. Contemp. Math. **426**, 339-404 (2007)
91. Triggiani, R., Yao, P.F.: Inverse/observability estimates for the Schrödinger equations with variable coefficients. Control Cybern. **28**, no. 3, 627-664 (1999)
92. Triggiani, R., Yao, P.F.: Carleman estimates with no lower order terms for general Riemann wave equations. Global uniqueness and observability in one shot. Appl. Math. Optim. **46**, no. 2-3, 331-375 (2002)
93. Washburn, D.: A semigroup theoretic approach to modeling of boundary input problems. In: Proc. IFIP Work. Conf. (University of Rome, Italy), LNCIS, Springer-Verlag (1977)
94. Washburn, D.: A bound on the boundary input map for parabolic equations with applications to time optimal control. SIAM J.Control **17**, 652-671 (1979)
95. Yao, P.F.: On the observability inequalities for exact controllability of wave equations with variable coefficients. SIAM J. Control Optim. **37**, 1568-1599 (1999)
96. Yao, P.F.: Observability inequalities for the Euler–Bernoulli plates with variable coefficients. Contemp. Math. **268**, 383-406 (2000)
97. Yao, P.F.: Observability inequalities for shallow shells. SIAM J. Control Optim. 38 (2000), 1729-1756

# Uniform Asymptotics of Green's Kernels for Mixed and Neumann Problems in Domains with Small Holes and Inclusions

Vladimir Maz'ya and Alexander Movchan

*To the memory of S.L. Sobolev*

**Abstract** Uniform asymptotic approximations of Green's kernels for the harmonic mixed and Neumann boundary value problems in domains with singularly perturbed boundaries are obtained. We consider domains with small holes (in particular, cracks) or inclusions. Formal asymptotic algorithms are supplied with rigorous estimates of the remainder terms.

## 1 Introduction

There is a wide range of applications in physics and structural mechanics involving perforated domains and bodies with defects of different types. Direct numerical treatment of such problems is sometimes inefficient, especially for situations where the right-hand sides in the equations and/or boundary conditions have singularities. Asymptotic approximations are important for problems of this kind and sometimes can be directly incorporated into computational algorithms if desirable.

Asymptotic formulas for Green's kernels of several classical boundary value problems under small variations of a domain were obtained in the pioneering paper [2] by Hadamard. These asymptotic approximations are related to the case of a *regularly perturbed domain*, when the boundary $\partial\Omega_\varepsilon$ of the perturbed

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series*.
doi: 10.1007/978-0-387-85652-0, 

domain approximates the limit boundary $\partial\Omega$ in such a way that the angle between the outward normals at nearby points of $\partial\Omega$ and $\partial\Omega_\varepsilon$ is small.

Asymptotic approximations in [2] are not uniform with respect to the independent variables. Results on uniform asymptotic approximations of Green's kernels in various *singularly perturbed domains* are formulated in [5]. Detailed derivation and analysis of uniform asymptotic formulas for Green's functions of the *Dirichlet problem* for the operator $-\Delta$ in $n$-dimensional domains with small holes are given in [6]. In particular, the asymptotic approximation, obtained in [6], for Green's function of the Dirichlet problem in a two-dimensional domain $\Omega_\varepsilon$ with an inclusion $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$ has the form

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + g(\boldsymbol{\xi},\boldsymbol{\eta}) + g(\boldsymbol{\xi},\infty) + g(\infty,\boldsymbol{\eta}) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|}{r_F} \\ &\quad - \frac{2\pi}{\log(\varepsilon r_F {R_\Omega}^{-1})}\left(G(\mathbf{x},0) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\xi}|}{r_F} - g(\boldsymbol{\xi},\infty)\right) \\ &\quad \times\left(G(0,\mathbf{y}) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\eta}|}{r_F} - g(\infty,\boldsymbol{\eta})\right) + O(\varepsilon), \end{aligned} \tag{1.1}$$

where $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$, $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$, $G$ and $g$ are Green's functions of "model" interior and exterior Dirichlet problems in "limit" domains $\Omega$ and $\mathbf{R}^2 \setminus F$, independent of $\varepsilon$; $R_\Omega$ and $r_F$ are the inner (with respect to $\mathbf{O}$) and outer conformal radii of $\Omega$ and $F$ respectively (see [8, Appendix G]).

Approximations of this type are readily applicable to numerical simulations. For example, in Fig. 1 we show the regular part of Green's function $G_\varepsilon$ in a two-dimensional domain with a small circular inclusion. The results on two diagrams are practically indistinguishable, while in Fig. 1a the data are obtained via the uniform asymptotic approximation, whereas Fig. 1b presents the result of independent finite element computations produced in COMSOL (courtesy of Dr. M. Nieves).

The aim of the present paper is to derive and justify asymptotic approximations of Green's kernels for singularly perturbed domains whose boundary, or some part of it, supports the *Neumann boundary condition.* Although the corresponding asymptotic formulas to be obtained and (1.1) are of similar nature, the former have some new features and require individual treatments. We also derive simpler asymptotic formulas, which become efficient when certain constraints are imposed on the independent variables.

Sections 2 and 3 deal with the Dirichlet–Neumann problems in two-dimensional domains with small holes, inclusions or cracks. Section 4 gives the uniform approximation of Green's function for the Neumann problem in a domain of the same type. Finally, in Sect. 5 we formulate similar asymptotic approximations of Green's kernels in three-dimensional domains with small holes or small inclusions.

(a)

(b)

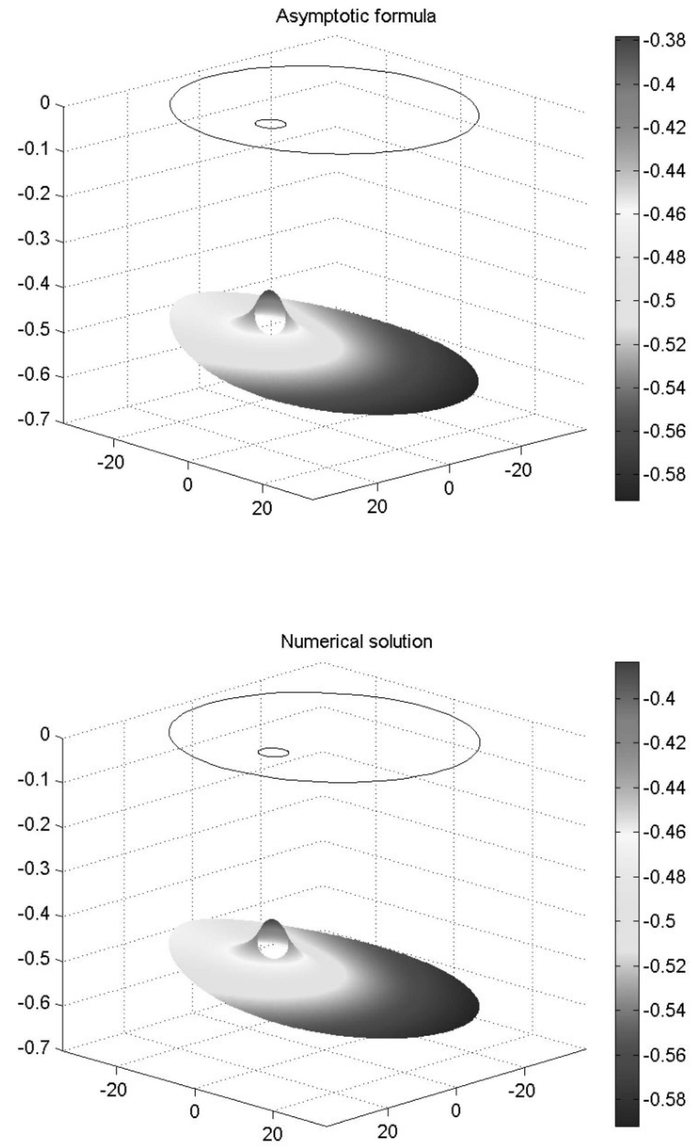


**Fig. 1** (a) Regular part of Green's function, computed via the asymptotic formula (1.1). (b) A finite element computation (in COMSOL) for the regular part of Green's function.

# 2 Green's Kernel for a Mixed Boundary Value Problem in a Planar Domain with a Small Hole or a Crack

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, which contains the origin $\mathbf{O}$, and let $F$ be a compact set in $\mathbb{R}^2$, $\mathbf{O} \in F$. We suppose that the boundary $\partial\Omega$ is smooth. This constraint is not essential and can be considerably weakened. We assume, without loss of generality, that diam $F = 1/2$ and $\text{dist}(\mathbf{O}, \partial\Omega) = 1$. We also introduce the set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$, with $\varepsilon$ being a small positive parameter. The boundary $\partial F$ is required to be piecewise smooth, with the angle openings from the side of $\mathbb{R}^2 \setminus F$ belonging to $(0, 2\pi]$. In the case of a crack, $\partial F$ and $\partial F_\varepsilon$ are treated as two-sided. We assume that $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ is connected, and in the sequel we refer to it as a domain with a small hole (or possibly a small crack).

Let $G_\varepsilon^{(N)}$ denote Green's function of the operator $-\Delta$ with the Neumann data on $\partial F_\varepsilon$ and the Dirichlet data on $\partial\Omega$. In other words, $G_\varepsilon^{(N)}$ is a solution of the problem

$$\Delta_x G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon, \tag{2.1}$$

$$G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon, \tag{2.2}$$

$$\frac{\partial G_\varepsilon^{(N)}}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon. \tag{2.3}$$

Here and elsewhere, *the Neumann condition is understood in the variational sense.*

In this section, we construct an asymptotic approximation of $G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y})$, uniform with respect to $\mathbf{x}$ and $\mathbf{y}$ in $\Omega_\varepsilon$.

## *2.1 Special solutions of model problems*

While constructing the asymptotic approximation of $G_\varepsilon^{(N)}$, we use the variational solutions $G(\mathbf{x}, \mathbf{y}), \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}), \zeta(\varepsilon^{-1}\mathbf{x})$ and $\mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$ of certain model problems in the limit domains $\Omega$ and $\mathbb{R}^2 \setminus F$. It is standard that all solutions, introduced in this subsection, exist and are unique. We describe these solutions.

1. Let $G$ be *Green's function for the Dirichlet problem in* $\Omega$:

$$G(\mathbf{x}, \mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1} - H(\mathbf{x}, \mathbf{y}), \tag{2.1}$$

where $H$ is the regular part of $G$, i.e., a unique solution of the Dirichlet problem

$$\Delta_x H(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega, \tag{2.2}$$

$$H(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1}, \quad \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega. \tag{2.3}$$

2. We introduce the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. The notation $\zeta$ is used for a unique special solution of the Dirichlet problem:

$$\Delta\zeta(\boldsymbol{\xi}) = 0 \ \text{ in } \ \mathbb{R}^2\setminus F, \tag{2.4}$$

$$\zeta(\boldsymbol{\xi}) = 0 \ \text{ for } \ \boldsymbol{\xi}\in\partial F, \tag{2.5}$$

$$\zeta(\boldsymbol{\xi}) = (2\pi)^{-1}\log|\boldsymbol{\xi}| + \zeta_\infty + O(|\boldsymbol{\xi}|^{-1}) \ \text{ as } |\boldsymbol{\xi}|\to\infty, \tag{2.6}$$

where $\zeta_\infty$ is constant.

Also, it can be shown that $\zeta$ is the limit of Green's function $\mathcal{G}$ of the exterior Dirichlet problem in $\mathbb{R}^2\setminus F$

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}|\to\infty}\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{2.7}$$

where

$$\Delta_\xi\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^2\setminus F, \tag{2.8}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi}\in\partial F,\ \boldsymbol{\eta}\in\mathbb{R}^2\setminus F, \tag{2.9}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) \ \text{ is bounded as } |\boldsymbol{\xi}|\to\infty \text{ and } \boldsymbol{\eta}\in\mathbb{R}^2\setminus F. \tag{2.10}$$

The representation (2.7) follows from Green's formula applied to $\zeta$ and $\mathcal{G}$. Here and elsewhere, $B_R = \{\mathbf{X}\in\mathbb{R}^2 :\ |\mathbf{X}| < R\}$. We derive

$$\begin{aligned}\zeta(\boldsymbol{\eta}) &= -\lim_{R\to\infty}\int\limits_{B_R\setminus F}\zeta(\boldsymbol{\xi})\Delta_\xi\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})d\boldsymbol{\xi}\\ &= \lim_{R\to\infty}\int\limits_{|\boldsymbol{\xi}|=R}\Big(\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})\frac{\partial\zeta(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - \zeta(\boldsymbol{\xi})\frac{\partial\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})}{\partial|\boldsymbol{\xi}|}\Big)dS_\xi\\ &= (2\pi)^{-1}\lim_{R\to\infty}\int\limits_{|\boldsymbol{\xi}|=R}\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})|\boldsymbol{\xi}|^{-1}dS_\xi = \mathcal{G}(\infty,\boldsymbol{\eta}),\end{aligned} \tag{2.11}$$

which yields (2.7).

3. Let $\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta})$ be *the Neumann function* in $\mathbb{R}^2\setminus F$ defined by

$$\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta}) = (2\pi)^{-1}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - h_N(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{2.12}$$

where $h_N$ is the regular part of $\mathcal{N}$ subject to

$$\Delta_\xi h_N(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^2\setminus F, \tag{2.13}$$

$$\frac{\partial h_N}{\partial n_\xi}(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{1}{2\pi}\frac{\partial}{\partial n_\xi}(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}), \quad \boldsymbol{\xi}\in\partial F,\ \boldsymbol{\eta}\in\mathbb{R}^2\setminus F, \tag{2.14}$$

$$h_N(\boldsymbol{\xi},\boldsymbol{\eta})\to 0 \ \text{ as } |\boldsymbol{\xi}|\to\infty,\ \boldsymbol{\eta}\in\mathbb{R}^2\setminus F. \tag{2.15}$$

We note that the Neumann function $\mathcal{N}$ used here, is symmetric. This follows from Green's formula applied to $U(\mathbf{X}) := \mathcal{N}(\mathbf{X},\boldsymbol{\xi})$ and $V(\mathbf{X}) := \mathcal{N}(\mathbf{x},\boldsymbol{\eta})$, where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are arbitrary fixed points in $\mathbb{R}^2\setminus F$. We have

$$U(\boldsymbol{\eta}) - V(\boldsymbol{\xi}) = \lim_{R\to\infty}\int\limits_{B_R\setminus F}\Big\{V(\mathbf{X})\Delta_x U(\mathbf{X}) - U(\mathbf{X})\Delta_X V(\mathbf{X})\Big\}d\mathbf{X}$$

$$= \lim_{R\to\infty}\int\limits_{|\mathbf{X}|=R}\{V(\mathbf{X})\frac{\partial}{\partial|\mathbf{X}|}U(\mathbf{X}) - U(\mathbf{X})\frac{\partial}{\partial|\mathbf{X}|}V(\mathbf{X})\}dS_X$$

$$= -\lim_{R\to\infty}(4\pi^2R)^{-1}\int\limits_{|\mathbf{X}|=R}\Big\{(\log|\mathbf{X}-\boldsymbol{\eta}|^{-1}+O(R^{-1}))\Big(\frac{\mathbf{X}\cdot(\mathbf{X}-\boldsymbol{\xi})}{|\mathbf{X}-\boldsymbol{\xi}|^2}+O(R^{-2})\Big)$$

$$-(\log|\mathbf{X}-\boldsymbol{\xi}|^{-1}+O(R^{-1}))\Big(\frac{\mathbf{X}\cdot(\mathbf{X}-\boldsymbol{\eta})}{|\mathbf{X}-\boldsymbol{\eta}|^2}+O(R^{-2})\Big)\Big\}dS_x = 0.$$

Thus,

$$0 = U(\boldsymbol{\eta}) - V(\boldsymbol{\xi}) = \mathcal{N}(\boldsymbol{\eta},\boldsymbol{\xi}) - \mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta}).$$

4. The *vector of dipole fields* $\boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = (\mathcal{D}_1(\boldsymbol{\xi}),\mathcal{D}_2(\boldsymbol{\xi}))^T$ is a solution of the exterior Neumann problem

$$\Delta\boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = 0 \ \text{ in } \ \mathbb{R}^2\setminus F, \tag{2.16}$$

$$\frac{\partial\mathcal{D}_j}{\partial n}(\boldsymbol{\xi}) = n_j \ \text{ for } \ \boldsymbol{\xi}\in\partial F,\ j=1,2, \tag{2.17}$$

$$\mathcal{D}_j(\boldsymbol{\xi})\to 0 \ \text{ as } \ |\boldsymbol{\xi}|\to\infty,\ j=1,2, \tag{2.18}$$

where $n_1$ and $n_2$ are components of the unit normal on $\partial F$.

## 2.2 The dipole matrix $\mathcal{P}$

The dipole fields $\mathcal{D}_j, j=1,2$, defined in (2.16)–(2.18), allow for the asymptotic representation (see, for example, [8])

$$\mathcal{D}_j(\boldsymbol{\xi}) = \frac{1}{2\pi}\sum_{k=1}^{2}\frac{\mathcal{P}_{jk}\xi_k}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-2}), \tag{2.1}$$

where $|\boldsymbol{\xi}| > 2$, and $\boldsymbol{\mathcal{P}} = (\mathcal{P}_{jk})_{j,k=1}^2$ is the *dipole matrix.*

The symmetry of $\boldsymbol{\mathcal{P}}$ can be verified as follows. Let $B_R$ be a disk of sufficiently large radius $R$ centered at the origin. We apply Green's formula to $\xi_j - \mathcal{D}_j(\boldsymbol{\xi})$ and $\mathcal{D}_k(\boldsymbol{\xi})$ in $B_R \setminus F$, and deduce

$$\int\limits_{\partial B_R} \Big\{(\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\frac{\partial D_k(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi})\frac{\partial}{\partial|\boldsymbol{\xi}|}(\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\Big\} dS$$

$$= -\int\limits_{\partial F} (\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\frac{\partial \mathcal{D}_k(\boldsymbol{\xi})}{\partial n} dS, \tag{2.2}$$

where $\partial/\partial n$ is the normal derivative in the direction of the interior normal with respect to $F$. In the limit, as $R \to \infty$, the integral on the left-hand side of (2.2) tends to $-\mathcal{P}_{kj}$, whereas the integral on the right-hand side becomes

$$-\int\limits_{\partial F} \xi_j \frac{\partial \xi_k}{\partial n} dS + \int\limits_{\partial F} \mathcal{D}_j(\boldsymbol{\xi})\frac{\partial \mathcal{D}_k(\boldsymbol{\xi})}{\partial n} dS$$

$$= \delta_{jk}\text{meas}(F) + \int\limits_{\mathbb{R}^2\setminus F} \nabla\mathcal{D}_j(\boldsymbol{\xi}) \cdot \nabla\mathcal{D}_k(\boldsymbol{\xi})\ d\boldsymbol{\xi},$$

where meas$(F)$ stands for the two-dimensional Lebesgue measure of the set $F$. Thus, the representation for components of the dipole matrix takes the form

$$P_{kj} = -\delta_{jk}\text{meas}(F) - \int\limits_{\mathbb{R}^2\setminus F} \nabla\mathcal{D}_j(\boldsymbol{\xi}) \cdot \nabla\mathcal{D}_k(\boldsymbol{\xi})\ d\boldsymbol{\xi}, \tag{2.3}$$

which implies that the *dipole matrix* $\boldsymbol{\mathcal{P}}$ *for the hole* $F$ *is symmetric and negative definite.*

## *2.3 Pointwise estimate of a solution to the exterior Neumann problem*

In this subsection, we make use of the function spaces $L_2^1(\mathbb{R}^2 \setminus F)$, $W_p^1(\mathbb{R}^2 \setminus F)$ and $W_p^{-1/p}(\partial F)$. The first of them is the space of distributions whose gradients belong to $L_2(\mathbb{R}^2 \setminus F)$. The second one is the usual Sobolev space consisting of functions in $L_p(\mathbb{R}^2 \setminus F)$ with distributional first derivatives in $L_p(\mathbb{R}^2 \setminus F)$. Finally, $W_p^{-1/p}(\partial F)$ stands for the dual of the space of traces on $\partial F$ of functions in $W_{p'}^1(\mathbb{R}^2 \setminus F)$, $p + p' = pp'$.

The following pointwise estimate will be used repeatedly in the sequel.

**Lemma 2.1.** *Let $U \in L^1_2(\mathbb{R}^2 \setminus F)$ be a solution of the exterior Neumann problem*

$$\Delta U(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, \tag{2.1}$$

$$\frac{\partial U}{\partial n}(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \partial F, \tag{2.2}$$

$$U(\boldsymbol{\xi}) \to 0 \quad as \quad |\boldsymbol{\xi}| \to \infty, \tag{2.3}$$

*where $\partial/\partial n$ is the normal derivative on $\partial F$, outward with respect to $\mathbb{R}^2 \setminus F$, and $\varphi \in L_\infty(\partial F)$,*

$$\int\limits_{\partial F} \varphi(\boldsymbol{\xi}) ds_\xi = 0. \tag{2.4}$$

*We also assume that*

$$\int\limits_{\partial F} U(\boldsymbol{\xi}) \frac{\partial \zeta}{\partial n}(\boldsymbol{\xi}) ds_\xi = 0, \tag{2.5}$$

*where $\zeta$ is the same as in* (2.7). *Then*

$$\sup_{\boldsymbol{\xi} \in \mathbb{R}^2 \setminus F} \{(|\boldsymbol{\xi}| + 1)|U(\boldsymbol{\xi})|\} \leqslant C \|\varphi\|_{L_\infty(\partial F)}, \tag{2.6}$$

*where $C$ is a constant depending on $\partial F$.*

*Proof.* Let $B_r$ denote the disk of radius $r$ centered at $\mathbf{O}$, and let $W^1_2(B_r \setminus F)$ be the space of restrictions of functions in $W^1_2(\mathbb{R}^2 \setminus F)$ to $B_r \setminus F$. By the $W^1_p$ local coercivity result [7], $U \in W^1_p(B_2 \setminus F)$ for any $p \in (1, 4)$, and

$$\|U\|_{W^1_p(B_2 \setminus F)} \leqslant C\Big(\|\varphi\|_{W_p^{-1/p}(\partial F)} + \|U\|_{L_2(B_3 \setminus F)}\Big). \tag{2.7}$$

The first term on the right-hand side of (2.7) satisfies

$$\|\varphi\|_{W_p^{-1/p}(\partial F)} \leqslant C \|\varphi\|_{L_\infty(\partial F)}. \tag{2.8}$$

From (2.1) and (2.2) it follows that

$$\|\nabla U\|^2_{L_2(\mathbb{R}^2 \setminus F)} = \int\limits_{\partial F} U(\boldsymbol{\xi}) \varphi(\boldsymbol{\xi}) dS \leqslant \|U\|_{L_2(\partial F)} \|\varphi\|_{L_2(\partial F)}. \tag{2.9}$$

Note that, by the Sobolev trace theorem,

$$\|U\|_{L_q(\partial F)} \leqslant C \|U\|_{W^1_2(B_2 \setminus F)} \tag{2.10}$$

for any $q < \infty$ (see, for instance, [4, Theorem 1.4.5]). From our assumptions on $F$ it follows that

$$\Big|\frac{\partial \zeta(\boldsymbol{\xi})}{\partial n}\Big| \leqslant C(\delta(\boldsymbol{\xi}))^{-1/2}, \tag{2.11}$$

where $\delta(\boldsymbol{\xi})$ is the distance from $\boldsymbol{\xi} \in \partial F$ to the nearest angle vertex on $\partial F$. Hence

$$\Big| \int\limits_{\partial F} U(\boldsymbol{\xi}) \frac{\partial \zeta(\boldsymbol{\xi})}{\partial n} dS \Big| \leqslant C \|U\|_{L_q(\partial F)} \tag{2.12}$$

for any $q > 2$. This inequality, together with (2.10), shows that the left-hand side in (2.12) is a semi-norm, continuous in $W_2^1(B_2 \setminus F)$. Moreover,

$$\int\limits_{\partial F} \frac{\partial \zeta}{\partial n}(\boldsymbol{\xi}) dS = \lim_{R \to \infty} (2\pi)^{-1} \int\limits_{|\boldsymbol{\xi}|=R} \frac{\partial}{\partial |\boldsymbol{\xi}|} \log |\boldsymbol{\xi}| \; dS = 1.$$

Now, the Sobolev equivalent normalizations theorem [4, Sect. 1.1.15] implies that the norm in $W_2^1(B_2 \setminus F)$ is equivalent to the norm

$$\|\nabla U\|_{L_2(B_2 \setminus F)} + \Big| \int\limits_{\partial F} U(\boldsymbol{\xi}) \frac{\partial \zeta}{\partial n}(\boldsymbol{\xi}) dS \Big|.$$

Combining this fact with (2.10) and using (2.5), we arrive at

$$\|U\|_{L_2(\partial F)} \leqslant C \|\nabla U\|_{L_2(\mathbb{R}^2 \setminus F)}. \tag{2.13}$$

Then, (2.9) and (2.13) yield

$$\|\nabla U\|_{L_2(\mathbb{R}^2 \setminus F)} + \|U\|_{L_2(\partial F)} \leqslant C \|\varphi\|_{L_2(\partial F)}. \tag{2.14}$$

By (2.10), the norm in $W_2^1(B_3 \setminus F)$ is equivalent to the norm

$$\|\nabla U\|_{L_2(B_3 \setminus F)} + \|U\|_{L_2(\partial F)}.$$

Hence

$$\|U\|_{L_2(B_3 \setminus F)} \leqslant C \Big( \|\nabla U\|_{L_2(\mathbb{R}^2 \setminus F)} + \|U\|_{L_2(\partial F)} \Big), \tag{2.15}$$

which, together with (2.14), gives

$$\|U\|_{L_2(B_3 \setminus F)} \leqslant C \|\varphi\|_{L_2(\partial F)}. \tag{2.16}$$

Substituting the estimates (2.8) and (2.16) into (2.7), we arrive at

$$\|U\|_{W_p^1(B_2 \setminus F)} \leqslant C \|\varphi\|_{L_\infty(\partial F)}. \tag{2.17}$$

Recalling that $W_p^1(B_2 \setminus F)$ is embedded into $C(\overline{B_2 \setminus F})$ for $p > 2$, by another Sobolev theorem (see [4, Theorem 1.4.5]), we obtain

$$\sup_{B_2 \setminus F} |U| \leqslant C \|\varphi\|_{L_\infty(\partial F)}. \tag{2.18}$$

Since $U(\boldsymbol{\xi}) \to 0$ as $|\boldsymbol{\xi}| \to \infty$ (see (2.4) and (2.5)), we have the Poisson formula

$$U(\boldsymbol{\xi}) = \frac{1}{\pi} \Re \int_0^{2\pi} \frac{U(1,\theta')}{\rho e^{i(\theta-\theta')} - 1} d\theta', \quad \boldsymbol{\xi} = \rho e^{i\theta}, \tag{2.19}$$

which, together with (2.18), implies for $|\boldsymbol{\xi}| > 1$ that

$$\big(1 + |\boldsymbol{\xi}|\big)|U(\boldsymbol{\xi})| \leqslant C \max_{\boldsymbol{\xi}\in\partial B_1} |U(\boldsymbol{\xi})| \leqslant C\|\varphi\|_{L_\infty(\partial\omega)}. \tag{2.20}$$

Applying (2.18) once more, we complete the proof. □

## 2.4 Asymptotic properties of the regular part of the Neumann function in $\mathbb{R}^2 \setminus F$

Lemma 2.1 proved in the previous section enables one to describe the asymptotic behavior of the function $h_N$ defined in (2.13)–(2.15).

**Lemma 2.1.** *The solution $h_N(\boldsymbol{\xi},\boldsymbol{\eta})$ of the problem* (2.13)–(2.15) *satisfies the estimate*

$$\Big|h_N(\boldsymbol{\xi},\boldsymbol{\eta}) - \frac{\boldsymbol{\mathcal{D}}(\boldsymbol{\eta})\cdot\boldsymbol{\xi}}{2\pi|\boldsymbol{\xi}|^2}\Big| \leqslant \text{Const}\ \ (1+|\boldsymbol{\eta}|)^{-1}|\boldsymbol{\xi}|^{-2} \tag{2.1}$$

*as $|\boldsymbol{\xi}| > 2$ and $\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F$.*

*Proof.* The leading-order approximation of the harmonic function $h_N(\boldsymbol{\xi},\boldsymbol{\eta})$, as $|\boldsymbol{\xi}| \to \infty$, is sought in the form

$$(2\pi)^{-1}|\boldsymbol{\xi}|^{-2}(C_1\xi_1 + C_2\xi_2).$$

Applying Green's formula in $B_R \setminus F$ to $h_N(\boldsymbol{\xi},\boldsymbol{\eta})$ and $\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j$, and taking the limit, as $R \to \infty$, we obtain

$$\lim_{R\to\infty} \int_{|\mathbf{x}|=R} \Big\{h_N(\boldsymbol{\xi})\frac{\partial(\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j)}{\partial|\boldsymbol{\xi}|} + (\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\frac{\partial h_N(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|}\Big\} dS_\xi$$

$$= \int_{\partial F} (\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j)\frac{\partial h_N(\boldsymbol{\xi})}{\partial n} dS_\xi, \tag{2.2}$$

where $\partial/\partial n$ is the normal derivative in the direction of the inward normal with respect to $F$. As $R \to \infty$, the left-hand side of (2.2) becomes

$$\frac{1}{2\pi} \lim_{R\to+\infty} \int_{|\mathbf{x}|=R} \Big\{ - 2\frac{(C_1\xi_1 + C_2\xi_2)\xi_j}{R^3}\Big\} dS_\xi$$

$$= -\frac{1}{\pi} \lim_{R\to+\infty} \int_0^{2\pi} (C_1 \cos\theta + C_2 \sin\theta) R^{-1} \xi_j d\theta = -C_j. \qquad (2.3)$$

Taking into account the definition of the dipole fields $\mathcal{D}_j$ (see (2.16)–(2.18)) and the definition of the regular part $h_N$ of the Neumann function (see (2.13)–(2.15)) in $\mathbb{R}^2 \setminus F$, we can reduce the integral $\mathcal{I}$ on the right-hand side of (2.2) to the form

$$\mathcal{I} = \frac{1}{2\pi}\Big\{ \int_{\partial F} \Big( \mathcal{D}_j(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi}\Big( \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big) - \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} \frac{\partial}{\partial n_\xi}\mathcal{D}_j(\boldsymbol{\xi})\Big) dS_\xi + \int_{\partial F} \Big( n_j \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - \xi_j \frac{\partial}{\partial n_\xi}\Big(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big)\Big) dS_\xi \Big\}. \qquad (2.4)$$

The second integral in (2.4) equals zero. Applying Green's formula to the first integral in (2.4), we obtain

$$\frac{1}{2\pi} \int_{\partial F} \Big( \mathcal{D}_j(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi}\Big( \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big) - \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} \frac{\partial}{\partial n_\xi}\mathcal{D}_j(\boldsymbol{\xi})\Big) dS_\xi = -\mathcal{D}_j(\boldsymbol{\eta}). \qquad (2.5)$$

Hence from (2.3)–(2.5) it follows that

$$C_j = \mathcal{D}_j(\boldsymbol{\eta}),\ j = 1, 2. \qquad (2.6)$$

We note that the function

$$h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\eta}) \cdot \nabla_\xi \Big( \frac{1}{2\pi} \log|\boldsymbol{\xi}|^{-1}\Big) \qquad (2.7)$$

is harmonic in $\mathbb{R}^2 \setminus F$, both in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, and it vanishes at infinity. Using (2.17) and (2.14), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial n_\eta}\Big( h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\eta}) \cdot \nabla_\xi \Big( \frac{1}{2\pi} \log|\boldsymbol{\xi}|^{-1}\Big)\Big) \\
&\quad = \frac{\partial}{\partial n_\eta} h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) + \mathbf{n} \cdot \nabla_\xi \Big( \frac{1}{2\pi} \log|\boldsymbol{\xi}|^{-1}\Big) \\
&\quad = -\mathbf{n} \cdot \nabla_\xi \Big\{ \frac{1}{2\pi} \log(|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1})\Big\} \\
&\quad = -\frac{1}{2\pi|\boldsymbol{\xi}|^2} \mathbf{n} \cdot \Big\{ \boldsymbol{\eta} - \frac{2\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^{-1})\Big\} \qquad (2.8)
\end{aligned}$$

as $\boldsymbol{\eta} \in \partial F$ and $|\boldsymbol{\xi}| > 2$. We also note that

$$\int\limits_{\partial F} \frac{\partial}{\partial n_\eta}\Big(h_N(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\eta}) \cdot \nabla_\xi (\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1})\Big) dS_\eta = 0.$$

Consider the problem (2.1)–(2.3) in the formulation of Lemma 2.1, where the variable $\boldsymbol{\xi}$ is replaced by $\boldsymbol{\eta}$, the differentiation is taken with respect to components of $\boldsymbol{\eta}$, and the function $U$ is changed for (2.7), with fixed $\boldsymbol{\xi}$. In this case, the right-hand side $\varphi$ in (2.2) is replaced by

$$\frac{\partial}{\partial n_\eta} h_N(\boldsymbol{\xi},\boldsymbol{\eta}) + \mathbf{n} \cdot \nabla_\xi \Big(\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1}\Big).$$

Then using (2.8) and applying Lemma 2.1, we obtain (2.1). □

Using the notion of the dipole matrix, from (2.1) and Lemma 2.1. we derive the following asymptotic representation of $h_N$.

**Corollary 2.1.** *Let $|\boldsymbol{\xi}| > 2$, and $|\boldsymbol{\eta}| > 2$. Then*

$$h_N(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{1}{4\pi^2}\sum_{j,k=1}^{2} \frac{\mathcal{P}_{jk}\xi_j\eta_k}{|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2} + O\Big(\frac{|\boldsymbol{\xi}|+|\boldsymbol{\eta}|}{|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2}\Big). \tag{2.9}$$

## *2.5 Maximum modulus estimate for solutions to the mixed problem in $\Omega_\varepsilon$ with the Neumann data on $\partial F_\varepsilon$*

In the sequel, when estimating the remainder term in the asymptotic representation of $G_\varepsilon(\mathbf{x},\mathbf{y})$, we use the following assertion.

**Lemma 2.1.** *Let $u$ be a function in $C(\overline{\Omega}_\varepsilon)$ such that $\nabla u$ is square integrable in a neighborhood of $\partial F_\varepsilon$. Also, let $u$ be a solution of the mixed boundary value problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{2.1}$$

$$u(\mathbf{x}) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{2.2}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{2.3}$$

*where $\varphi \in C(\partial\Omega)$, $\psi_\varepsilon \in L_\infty(\partial F_\varepsilon)$, and*

$$\int\limits_{\partial F_\varepsilon} \psi_\varepsilon(\mathbf{x}) ds = 0. \tag{2.4}$$

*Then there exists a positive constant* $C$*, independent of* $\varepsilon$ *and such that*

$$\|u\|_{C(\overline{\Omega}_\varepsilon)} \leqslant \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{2.5}$$

*Proof.* (a) We introduce the inverse operator

$$\mathfrak{N} : \psi \to v \tag{2.6}$$

for the boundary value problem

$$\Delta v(\boldsymbol{\xi}) = 0, \ \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, \tag{2.7}$$

$$\frac{\partial v}{\partial n}(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi}), \ \boldsymbol{\xi} \in \partial F, \tag{2.8}$$

$$v(\boldsymbol{\xi}) \to 0 \text{ as } |\boldsymbol{\xi}| \to \infty, \tag{2.9}$$

where $\psi \in L_\infty(\partial F)$, and

$$\int_{\partial F} \psi(\boldsymbol{\xi}) ds_\xi = 0. \tag{2.10}$$

In the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$, the operator $\mathfrak{N}_\varepsilon$ is defined by

$$(\mathfrak{N}_\varepsilon \psi_\varepsilon)(\mathbf{x}) = (\mathfrak{N}\psi)(\boldsymbol{\xi}), \tag{2.11}$$

where $\psi_\varepsilon(\mathbf{x}) = \varepsilon^{-1}\psi(\varepsilon^{-1}\mathbf{x})$.

(b) We look for the solution $u$ of (2.1)–(2.4) in the form

$$u = V(\mathbf{x}) + W(\mathbf{x}), \tag{2.12}$$

where $V = \mathfrak{N}_\varepsilon \psi_\varepsilon$, and the function $W$ satisfies the problem

$$\Delta W(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{2.13}$$

$$\frac{\partial W}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \tag{2.14}$$

$$W(\mathbf{x}) = \varphi(\mathbf{x}) - V(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{2.15}$$

By Lemma 2.1, we have

$$\max_{\overline{\Omega}_\varepsilon} |V| = \max_{\overline{\Omega}_\varepsilon} |\mathfrak{N}_\varepsilon \psi_\varepsilon| \leqslant \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{2.16}$$

Hence, as follows from (2.15) and (2.16),

$$\max_{\partial\Omega} |W| \leqslant \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}, \tag{2.17}$$

and, by the weak maximum principle for variational solutions (see, for example, [1, pp. 215–216]) of (2.13)–(2.15), we obtain

$$\max_{\overline{\Omega}_\varepsilon} |W| \leqslant \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{2.18}$$

The result follows from (2.16), (2.18) combined with (2.12). □

## 2.6 Approximation of Green's function $G_\varepsilon^{(N)}$

The required approximation of $G_\varepsilon^{(N)}$ is given in the next theorem.

**Theorem 2.1.** *Green's function* $G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y})$ *for the boundary value problem* (2.1)–(2.3) *with the Neumann data on* $\partial F_\varepsilon$ *and the Dirichlet data on* $\partial\Omega$ *has the asymptotic representation*

$$G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) + \mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1} \log(\varepsilon^{-1}|\mathbf{x} - \mathbf{y}|)$$

$$+\varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x H(0, \mathbf{y}) + \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y H(\mathbf{x}, 0) + r_\varepsilon(\mathbf{x}, \mathbf{y}), \tag{2.1}$$

*where*

$$|r_\varepsilon(\mathbf{x}, \mathbf{y})| \leqslant \text{Const } \varepsilon^2 \tag{2.2}$$

*uniformly with respect to* $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. *Here,* $G$, $\mathcal{N}$, $\boldsymbol{\mathcal{D}}$, *and* $H$ *are the same as in Sect.* 2.1.

*Proof.* We begin with the formal argument leading to (2.1). First, we note that

$$N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1} \log(\varepsilon^{-1}|\mathbf{x} - \mathbf{y}|) = -h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}),$$

and then represent $G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y})$ in the form

$$G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + \rho_\varepsilon(\mathbf{x}, \mathbf{y}). \tag{2.3}$$

By the direct substitution of (2.3) into (2.1)–(2.3) and using Lemma 2.1, we deduce that $\rho_\varepsilon(\mathbf{x}, \mathbf{y})$ satisfies the boundary value problem

$$\Delta_x \rho_\varepsilon(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon,$$

$$\rho_\varepsilon(\mathbf{x}, \mathbf{y}) = h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$$

$$= \frac{\varepsilon}{2\pi} \boldsymbol{\mathcal{D}}\Big(\frac{\mathbf{y}}{\varepsilon}\Big) \cdot \frac{\mathbf{x}}{|\mathbf{x}|^2} + O(\varepsilon^2) \quad \text{for } \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon, \tag{2.4}$$

and

$$\frac{\partial \rho_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial n_x} H(\mathbf{x},\mathbf{y})$$
$$= \mathbf{n}\cdot\nabla_x H(0,\mathbf{y}) + O(\varepsilon) \quad \text{for } \mathbf{x}\in\partial F_\varepsilon, \mathbf{y}\in\Omega_\varepsilon. \tag{2.5}$$

Hence, by (2.2), (2.3) and (2.16)–(2.18), the leading-order approximation of $\rho_\varepsilon$ is

$$\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0),$$

which, together with (2.3), leads to (2.1).

Now, we prove the remainder estimate (2.2). The direct substitution of (2.1) into (2.1)–(2.3) yields the boundary value problem for $r_\varepsilon$:

$$\Delta_x r_\varepsilon(\mathbf{x},\mathbf{y}) = 0 \quad \text{for } \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \tag{2.6}$$

$$r_\varepsilon(\mathbf{x},\mathbf{y}) = h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$
$$-\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0) \tag{2.7}$$
$$\text{for } \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon,$$

$$\frac{\partial r_\varepsilon(\mathbf{x},\mathbf{y})}{\partial n_x} = \mathbf{n}\cdot\nabla_x H(\mathbf{x},\mathbf{y}) - \varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})\Big)$$
$$-\varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)\Big) \tag{2.8}$$
$$\text{for } \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon.$$

We note that every term on the right-hand side of (2.8) has zero average on $\partial F_\varepsilon$, and hence

$$\int_{\partial F_\varepsilon} \frac{\partial r_\varepsilon(\mathbf{x},\mathbf{y})}{\partial n_x} dS_x = 0. \tag{2.9}$$

From Lemma 2.1 it follows that

$$|h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)| \leqslant \text{Const } \varepsilon^2 \tag{2.10}$$

uniformly with respect to $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$. Since $|\boldsymbol{\mathcal{D}}(\boldsymbol{\xi})| \leqslant \text{Const } |\boldsymbol{\xi}|^{-1}$, as $|\boldsymbol{\xi}|\to\infty$, and $\nabla_x H(0,\mathbf{y})$ is smooth on $\Omega_\varepsilon$, we deduce

$$|\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})| \leqslant \text{Const } \varepsilon^2 \tag{2.11}$$

uniformly with respect to $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$. By (2.10) and (2.11), the modulus of the right-hand side in (2.7) is bounded by Const $\varepsilon^2$ uniformly in $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$.

It also follows from the definition of the dipole fields $\mathcal{D}_j(\boldsymbol{\xi}), j = 1, 2$, and the smoothness of the function $H(\mathbf{x},\mathbf{y})$ for all $\mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon$ that

$$\left|\mathbf{n}\cdot\nabla_x H(\mathbf{x},\mathbf{y}) - \varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})\Big)\right| \leqslant \text{Const } \varepsilon \tag{2.12}$$

and

$$\left|\varepsilon \frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y H(\mathbf{x}, 0)\Big)\right| \leqslant \text{Const } \varepsilon \tag{2.13}$$

uniformly with respect to $\mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon$. These estimates imply that the modulus of the right-hand side in (2.8) is bounded by Const $\varepsilon$ uniformly in $\mathbf{x} \in \partial F_\varepsilon$ and $\mathbf{y} \in \Omega_\varepsilon$.

Using the estimates on $\partial F_\varepsilon$ and $\partial \Omega$, just obtained, together with the orthogonality condition (2.9), we deduce that the right-hand sides of the problem (2.6)–(2.8) satisfy the conditions of Lemma 2.1. Applying Lemma 2.1, we find that $\|r_\varepsilon\|_{L_\infty(\Omega_\varepsilon)}$ is dominated by Const $\varepsilon^2$, which completes the proof. □

## *2.7 Simpler asymptotic formulas for Green's function $G_\varepsilon^{(N)}$*

Here, we formulate two corollaries of Theorem 2.1. They contain simpler asymptotic formulas, which are efficient for the cases when both $\mathbf{x}$ and $\mathbf{y}$ are distant from $F_\varepsilon$ or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$.

**Corollary 2.1.** *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula holds*

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) &- \frac{\varepsilon^2}{4\pi^2} \frac{\mathbf{x}^T}{|\mathbf{x}|^2} \boldsymbol{\mathcal{P}} \frac{\mathbf{y}}{|\mathbf{y}|^2} \\ &+ \frac{\varepsilon^2}{2\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\nabla_x H(0, \mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^2}\boldsymbol{\mathcal{P}}\nabla_y H(\mathbf{x}, 0)\Big\} \\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2}), \end{aligned} \tag{2.1}$$

*where $H$ is the regular part of Green's function $G$ in $\Omega$, and $\boldsymbol{\mathcal{P}}$ is the dipole matrix for $F$, as defined in* (2.1).

*Proof.* Using (2.9) for the regular part $h_N$ of the Neumann function in $\mathbb{R}^2 \setminus F$, together with the asymptotic representation (2.1) of the dipole fields $\mathcal{D}_j$ in $\mathbb{R}^2 \setminus F$, we obtain

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) &- \frac{\varepsilon^2}{4\pi^2}\sum_{j,k=1}^{2} \frac{\mathcal{P}_{jk} x_j y_k}{|\mathbf{x}|^2|\mathbf{y}|^2} + O\Big(\varepsilon^3 \frac{|\mathbf{x}| + |\mathbf{y}|}{|\mathbf{x}|^2|\mathbf{y}|^2}\Big) \\ &+ \frac{1}{2\pi}\sum_{j,k=1}^{2}\Big\{\varepsilon^2 \mathcal{P}_{jk}\Big(\frac{x_k}{|\mathbf{x}|^2}\frac{\partial H}{\partial x_j}(0, \mathbf{y}) + \frac{y_k}{|\mathbf{y}|^2}\frac{\partial H}{\partial y_j}(\mathbf{x}, 0)\Big) \\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2})\Big\} + O(\varepsilon^2). \end{aligned} \tag{2.2}$$

Combining the remainder terms and adopting the matrix representation involving the dipole matrix $\boldsymbol{\mathcal{P}}$, we arrive at (2.1). □

Formula (2.1) becomes efficient when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently distant from the small hole $F_\varepsilon$. Compared to (2.1), formula (2.1) does not involve special solutions of model problems in $\mathbb{R}^2 \setminus F$, while the influence of the hole $F$ is seen through the dipole matrix $\boldsymbol{\mathcal{P}}$.

**Corollary 2.2.** *The following asymptotic formula for Green's function* $G_\varepsilon^{(N)}$ *of the boundary value problem* (2.1)–(2.3) *holds*:

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(0,0)$$
$$- (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x H(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y H(\mathbf{x},0)$$
$$+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2) \tag{2.3}$$

*for* $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (*Needless to say,* $\varepsilon^2$ *in the remainder can be omitted if the interior of* $F$ *is nonempty and contains the origin.*)

*Proof.* Using the Taylor expansion of $H(\mathbf{x},\mathbf{y})$ in a neighborhood of the origin, we obtain

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = -H(0,0) - \mathbf{x}\cdot\nabla_x H(0,\mathbf{y}) - \mathbf{y}\cdot\nabla_y H(\mathbf{y},0) + O(|\mathbf{x}|^2 + |\mathbf{y}|^2)$$
$$+ \mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (2\pi)^{-1}\log\varepsilon$$
$$+ \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})$$
$$+ \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0) + O(\varepsilon^2). \tag{2.4}$$

By substituting

$$\mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} + (2\pi)^{-1}\log\varepsilon - h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$

into (2.4) and rearranging the terms, we arrive at (2.3). □

## 3 Mixed Boundary Value Problem with the Dirichlet Condition on $\partial F_\varepsilon$

In the present section, the meaning of the notation $\Omega$, $F$, and $F_\varepsilon$, already used in Sect. 2, will be slightly altered. Hopefully, this will not lead to any confusion. Let $\Omega$ be a bounded domain with smooth boundary, and let $F$ stand for an arbitrary compact set in $\mathbb{R}^2$ of positive logarithmic capacity [3]. As in Sect. 2, it is assumed that diam $F = 1/2$, and $\mathrm{dist}(\mathbf{O},\partial\Omega) = 1$. We also set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$.

We consider the mixed boundary value problem in a two-dimensional domain $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ with the Dirichlet data on $\partial F_\varepsilon$ and the Neumann data on $\partial\Omega$.

Green's function $G_\varepsilon^{(D)}$ of this problem is a weak solution of

$$\Delta_x G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \tag{3.1}$$

$$G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \ \mathbf{y} \in \Omega_\varepsilon, \tag{3.2}$$

$$\frac{\partial G_\varepsilon^{(D)}}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon. \tag{3.3}$$

Before deriving an asymptotic approximation of $G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y})$, uniform with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$, we outline the properties of solutions of auxiliary model problems in limit domains.

## 3.1 Special solutions of model problems

1. Let $N(\mathbf{x},\mathbf{y})$ be the Neumann function in $\Omega$, i.e.,

$$\Delta N(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega, \tag{3.1}$$

$$\frac{\partial}{\partial n_x}\Big(N(\mathbf{x},\mathbf{y}) + (2\pi)^{-1}\log|\mathbf{x}|\Big) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega, \tag{3.2}$$

and

$$\int\limits_{\partial\Omega} N(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial n_x} \log|\mathbf{x}| ds_x = 0. \tag{3.3}$$

The condition (3.3) implies the symmetry of $N(\mathbf{x},\mathbf{y})$. In fact, let $U(\mathbf{x}) = N(\mathbf{x},\mathbf{z})$ and $V(\mathbf{x}) = N(\mathbf{x},\mathbf{y})$, where $\mathbf{z}$ and $\mathbf{y}$ are fixed points in $\Omega$. Then, applying Green's formula to $U$ and $V$ and using (3.1)–(3.3), we deduce

$$U(\mathbf{y}) - V(\mathbf{z}) = \int\limits_{\Omega} \Big(V(\mathbf{x})\Delta_x U(\mathbf{x}) - U(\mathbf{x})\Delta_x V(\mathbf{x})\Big) d\mathbf{x}$$

$$= \frac{1}{2\pi} \int\limits_{\partial\Omega} \Big(U(\mathbf{x})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - V(\mathbf{x})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|)\Big) dS_x$$

$$= \frac{1}{2\pi}\Big\{ \int\limits_{\partial\Omega} N(\mathbf{x},\mathbf{z})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) dS_x - \int\limits_{\partial\Omega} N(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|)\Big) dS_x \Big\} = 0,$$

where $\partial/\partial n_x$ is the normal derivative in the direction of the outward normal on $\partial\Omega$. Hence $N(\mathbf{y},\mathbf{z}) = N(\mathbf{z},\mathbf{y})$.

The regular part of the Neumann function is defined by

$$R(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x},\mathbf{y}). \tag{3.4}$$

Note that

$$R(0,\mathbf{y}) = -(2\pi)^{-2}\int\limits_{\partial\Omega}\log|\mathbf{x}|\frac{\partial}{\partial n}\log|\mathbf{x}|ds_x, \tag{3.5}$$

which is verified by applying Green's formula to $R(\mathbf{x},\mathbf{y})$ and $(2\pi)^{-1}\log|\mathbf{x}|$ as follows:

$$R(0,\mathbf{y}) = \frac{1}{2\pi}\int\limits_{\Omega} R(\mathbf{x},\mathbf{y})\Delta_x(\log|\mathbf{x}|)d\mathbf{x}$$

$$= \frac{1}{2\pi}\int\limits_{\partial\Omega}\Big(R(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - \log|\mathbf{x}|\frac{\partial}{\partial n_x}R(\mathbf{x},\mathbf{y})\Big)ds_x, \tag{3.6}$$

where $\partial/\partial n_x$ is the normal derivative in the outward direction on $\partial\Omega$. Taking into account (3.2), (3.3), and (3.4), we can write (3.6) in the form

$$R(0,\mathbf{y}) = \frac{1}{4\pi^2}\int\limits_{\partial\Omega}\Big(\log|\mathbf{x}-\mathbf{y}|^{-1}\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - \log|\mathbf{x}|\frac{\partial}{\partial n_x}(\log|\mathbf{x}-\mathbf{y}|^{-1})\Big)ds_x$$

$$+\frac{1}{2\pi}\int\limits_{\partial\Omega}\log|\mathbf{x}|\frac{\partial}{\partial n_x}(N(\mathbf{x},\mathbf{y}))ds_x. \tag{3.7}$$

The first integral in (3.7) is equal to zero, while the second integral in (3.7) is reduced to (3.5) because of the boundary condition (3.2).

As in Sect. 2, the notation $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ will be used for the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. The corresponding limit domain is $\mathbb{R}^2 \setminus F$.

2. Green's function $\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})$ for the Dirichlet problem in $\mathbb{R}^2 \setminus F$ is a unique solution to the problem (2.8)–(2.10). The regular part $h(\boldsymbol{\xi},\boldsymbol{\eta})$ of Green's function $\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})$ is

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = (2\pi)^{-1}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}). \tag{3.8}$$

3. Here and in the sequel, $\mathbf{D}(\boldsymbol{\xi})$ denotes a vector function, whose components $D_j,\ j=1,2$, satisfy the model problems

$$\Delta D_j(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, \tag{3.9}$$

$$D_j(\boldsymbol{\xi}) = \xi_j, \quad \boldsymbol{\xi} \in \partial F, \tag{3.10}$$

$$D_j(\boldsymbol{\xi}) \text{ is bounded as } |\boldsymbol{\xi}| \to \infty. \tag{3.11}$$

We use the notation $D_j^\infty = \lim_{|\boldsymbol{\xi}|\to\infty} D_j(\boldsymbol{\xi})$ and $\mathbf{D}^\infty = (D_1^\infty, D_2^\infty)^T$.

Application of Green's formula to $D_j$ and the function $\zeta$, defined in (2.4)–(2.6), gives

$$D_j^\infty = -\int\limits_{\partial F} \xi_j \frac{\zeta(\boldsymbol{\xi})}{\partial n} dS_\xi. \tag{3.12}$$

Here and in other derivations of this section, $\partial/\partial n$ on $\partial F$ is the normal derivative in the direction of the inward normal with respect to $F$.

We also find an additional connection between $D_j$ and $\zeta$ by analyzing the asymptotic formula (compare with (2.6))

$$\zeta(\boldsymbol{\xi}) = (2\pi)^{-1} \log |\boldsymbol{\xi}| + \zeta_\infty + \frac{1}{2\pi} \sum_{k=1}^{2} \frac{\alpha_k \xi_k}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-2}), \quad |\boldsymbol{\xi}| \to \infty, \tag{3.13}$$

and showing that

$$\alpha_k = -D_k^\infty. \tag{3.14}$$

Let us apply Green's formula to $\xi_j$ and $\zeta$:

$$\begin{aligned} \int\limits_{\partial F} \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial n} dS_\xi &= \int\limits_{\partial F} \Big\{ \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial n} - \zeta(\boldsymbol{\xi}) \frac{\partial \xi_j}{\partial n} \Big\} dS_\xi \\ &= -\lim_{R\to\infty} \int\limits_{|\boldsymbol{\xi}|=R} \Big\{ \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial |\boldsymbol{\xi}|} - \zeta(\boldsymbol{\xi}) \frac{\partial \xi_j}{\partial |\boldsymbol{\xi}|} \Big\} dS_\xi \\ &= \frac{1}{\pi} \lim_{R\to\infty} \int\limits_{|\boldsymbol{\xi}|=R} \sum_{k=1}^{2} \frac{\alpha_k \xi_k \xi_j}{|\boldsymbol{\xi}|^3} dS_\xi = \alpha_j. \end{aligned} \tag{3.15}$$

Then formulas (3.15) and (3.12) lead to (3.14).

## *3.2 Asymptotic property of the regular part of Green's function in* $\mathbb{R}^2 \setminus F$

Asymptotic representation at infinity for the regular part of Green's function in $\mathbb{R}^2 \setminus F$ is given by the following lemma.

**Lemma 3.1.** *The regular part* (3.8) *of* $\mathcal{G}$ *satisfies the estimate*

$$\Big| h(\boldsymbol{\xi}, \boldsymbol{\eta}) - (2\pi)^{-1} \log |\boldsymbol{\xi}|^{-1} + \zeta(\boldsymbol{\eta}) - \frac{1}{2\pi} \sum_{j=1}^{2} \frac{D_j(\boldsymbol{\eta}) \xi_j}{|\boldsymbol{\xi}|^2} \Big| \leqslant \frac{\text{Const}}{|\boldsymbol{\xi}|^2}, \tag{3.1}$$

*as* $|\boldsymbol{\xi}| > 2$, *and* $\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F$.

*Proof.* Let

$$\beta(\boldsymbol{\xi}, \boldsymbol{\eta}) = h(\boldsymbol{\xi}, \boldsymbol{\eta}) - (2\pi)^{-1} \log |\boldsymbol{\xi}|^{-1} + \zeta(\boldsymbol{\eta}) - \frac{1}{2\pi} \sum_{j=1}^{2} \frac{D_j(\boldsymbol{\eta})\xi_j}{|\boldsymbol{\xi}|^2}.$$

We have

$$\Delta_\eta \beta(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F,$$

and

$$\beta(\boldsymbol{\xi}, \boldsymbol{\eta}) = -\frac{1}{4\pi} \log \Big(1 - 2\frac{\boldsymbol{\xi} \cdot \boldsymbol{\eta}}{|\boldsymbol{\xi}|^2} + \frac{|\boldsymbol{\eta}|^2}{|\boldsymbol{\xi}|^2}\Big) - \frac{\boldsymbol{\xi} \cdot \boldsymbol{\eta}}{2\pi|\boldsymbol{\xi}|^2}$$

$$= -\frac{1}{4\pi|\boldsymbol{\xi}|^2} \Big\{ |\boldsymbol{\eta}|^2 - 2\frac{(\boldsymbol{\xi} \cdot \boldsymbol{\eta})^2}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-1}) \Big\} \tag{3.2}$$

for $\boldsymbol{\eta} \in \partial F$. By (2.4)–(2.6) and Green's formula,

$$\beta(\boldsymbol{\xi}, \infty) = -\int\limits_{\partial F} \beta(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial \zeta(\boldsymbol{\eta})}{\partial n_\eta} dS_\eta,$$

which, together with (3.2) and (2.11), implies

$$|\beta(\boldsymbol{\xi}, \infty)| \leqslant C \ |\boldsymbol{\xi}|^{-2}.$$

Hence the maximum principle gives (3.1). □

## *3.3 Maximum modulus estimate for solutions to the mixed problem in $\Omega_\varepsilon$ with the Dirichlet data on $\partial F_\varepsilon$*

**Lemma 3.1.** *Let $u$ be a function in $C(\overline{\Omega}_\varepsilon)$ such that $\nabla u$ is square integrable in a neighborhood of $\partial\Omega$. Let $u$ be a solution of the mixed problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{3.1}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3.2}$$

$$u(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{3.3}$$

*where $\psi \in C(\partial\Omega)$, $\varphi_\varepsilon \in C(\partial F_\varepsilon)$, and*

$$\int\limits_{\partial\Omega} \psi(\mathbf{x}) ds = 0. \tag{3.4}$$

*Then there exists a positive constant $C$ such that*

$$\|u\|_{C(\Omega_\varepsilon)} \leqslant \|\varphi_\varepsilon\|_{C(\partial F_\varepsilon)} + C\|\psi\|_{C(\partial\Omega)}. \tag{3.5}$$

*Proof.* (a) First, we introduce the inverse operator

$$\mathfrak{N}_\Omega : \psi \to w \tag{3.6}$$

for the interior Neumann problem in $\Omega$

$$\Delta w(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{3.7}$$

$$\frac{\partial w}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3.8}$$

with $\psi \in C(\partial\Omega)$ and

$$\int\limits_{\partial\Omega} \psi(\mathbf{x}) dS_x = 0 \quad \text{and} \quad \int\limits_{\partial\Omega} w(\mathbf{x}) \frac{\partial}{\partial n}\big(\log|\mathbf{x}|\big) dS_x = 0. \tag{3.9}$$

Applying Green's formula to $w(\mathbf{x})$ and $N(\mathbf{x}, \mathbf{y})$ in $\Omega$, we obtain

$$w(\mathbf{y}) = \int\limits_{\partial\Omega} \Big( N(\mathbf{x}, \mathbf{y})\psi(\mathbf{x}) + \frac{1}{2\pi} w(\mathbf{x}) \frac{\partial}{\partial n_x}(\log|\mathbf{x}|) \Big) dS_x.$$

Then the unique solution of (3.7)–(3.9) is given by

$$w(\mathbf{x}) = \int\limits_{\partial\Omega} N(\mathbf{x}, \mathbf{y})\psi(\mathbf{y}) dS_y, \tag{3.10}$$

and

$$\max_{\overline{\Omega}} |w| \leqslant C\|\psi\|_{C(\partial\Omega)}. \tag{3.11}$$

(b) The solution $u$ of (3.1)–(3.3) is sought in the form

$$u(\mathbf{x}) = w(\mathbf{x}) + v(\mathbf{x}), \tag{3.12}$$

where $w = \mathfrak{N}_\Omega \psi$ is defined by (3.10), whereas the second term $v$ satisfies the problem

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{3.13}$$

$$\frac{\partial v}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{3.14}$$

$$v(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}) - w(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon. \tag{3.15}$$

According to the estimate (3.11) and the maximum principle for variational solutions of (3.13)–(3.15) (see, for example, [1]), we have

$$\max_{\overline{\Omega}_\varepsilon} |v| \leqslant \|\varphi_\varepsilon\|_{C(\partial F_\varepsilon)} + C\|\psi\|_{C(\partial\Omega)}. \tag{3.16}$$

Finally, using the representation (3.12), together with the estimates (3.11) and (3.16), we obtain the result (3.5). This completes the proof. □

## 3.4 Approximation of Green's function $G_\varepsilon^{(D)}$

We give a uniform asymptotic formula for Green's function solving the problem (3.1)-(3.3).

**Theorem 3.1.** *Green's function* $G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y})$ *for problem* (3.1)–(3.3) *admits the asymptotic representation*

$$G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = \mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + N(\mathbf{x},\mathbf{y}) - (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0)$$
$$+\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + r_\varepsilon(\mathbf{x},\mathbf{y}), \tag{3.1}$$

*where* $\mathcal{G}, N, R, D$ *are defined in* (2.8)*–*(2.10)*,* (3.1)*–*(3.3)*,* (3.4)*,* (3.9)*–*(3.11)*, and*

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \leqslant \text{Const } \varepsilon^2,$$

*which is uniform with respect to* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$*.*

*Proof.* First, we describe the formal argument leading to (3.1). Let $\rho_\varepsilon(\mathbf{x},\mathbf{y}) = G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) - \mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$. This function satisfies the problem

$$\Delta_x\rho_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \tag{3.2}$$

$$\rho_\varepsilon(\mathbf{x},\mathbf{y}) = 0 \;\text{ when } \mathbf{x}\in\partial F_\varepsilon,\; \mathbf{y}\in\Omega_\varepsilon, \tag{3.3}$$

and

$$\begin{aligned}\frac{\partial\rho_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) &= -\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1} - h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\Big) \\ &= -\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x},\mathbf{y})\Big) \\ &\quad +\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}| + h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\Big),\end{aligned} \tag{3.4}$$

where $\mathbf{x}\in\partial\Omega,\;\mathbf{y}\in\Omega_\varepsilon$. Here, $h(\boldsymbol{\xi},\boldsymbol{\eta})$ is the regular part of Green's function $\mathcal{G}$ in $\mathbf{R}^2\setminus F$. Taking into account (3.4), we deduce that

$$\rho_\varepsilon(\mathbf{x},\mathbf{y}) = -R(\mathbf{x},\mathbf{y}) + R(0,0) + \mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y}), \tag{3.5}$$

where $R(\mathbf{x},\mathbf{y})$ is the regular part of the Neumann function $N(\mathbf{x},\mathbf{y})$ in $\Omega$, and $\mathcal{R}_\varepsilon$ is harmonic in $\Omega_\varepsilon$ and satisfies the boundary conditions

$$\frac{\partial \mathcal{R}_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}| + h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\Big) \text{ for } \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon, \quad (3.6)$$

$$\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\nabla_x R(0,\mathbf{y}) + O(\varepsilon^2) \text{ for } \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon. \quad (3.7)$$

The asymptotics of $h(\boldsymbol{\xi},\boldsymbol{\eta})$ given by Lemma 3.1, can be used in evaluation of the right-hand side in (3.6).

The boundary condition (3.7) can be written as

$$\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon\mathbf{D}(\boldsymbol{\xi})\cdot\nabla_x R(0,\mathbf{y}) = O(\varepsilon^2),$$

for $\mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon$. In turn, the boundary condition (3.6) is reduced to

$$\frac{\partial}{\partial n_x}\Big\{\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y}) - \varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)\Big\} = O(\varepsilon^2),$$

when $\mathbf{x}\in\partial\Omega,\ \ \mathbf{y}\in\Omega_\varepsilon$. Hence the representation (3.5) of $\rho_\varepsilon$ can be updated to the form

$$\begin{aligned}\rho_\varepsilon(\mathbf{x},\mathbf{y}) &= -R(\mathbf{x},\mathbf{y}) + R(0,0) \\ &+ \varepsilon\mathbf{D}(\boldsymbol{\xi})\cdot\nabla_x R(0,\mathbf{y}) + \varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0) + \mathcal{R}^{(1)}_\varepsilon(\mathbf{x},\mathbf{y}),\end{aligned} \quad (3.8)$$

where the principal part of $\mathcal{R}^{(1)}_\varepsilon(\mathbf{x},\mathbf{y})$ compensates for the leading term of the discrepancy $\varepsilon^2\boldsymbol{\xi}\cdot\nabla_x\big(\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)\big)\big|_{\mathbf{x}=0}$ brought by the term $\varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)$ into the boundary condition (3.3) on $\partial F_\varepsilon$. This leads to the required formula (3.1).

For the remainder $r_\varepsilon(\mathbf{x},\mathbf{y})$ in the asymptotic formula (3.1) we verify by the direct substitution that

$$\Delta_x r_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \quad (3.9)$$

and that the boundary condition (3.2) implies

$$r_\varepsilon(\mathbf{x},\mathbf{y}) = R(0,\mathbf{y}) - R(0,0) + \mathbf{x}\cdot\nabla_x R(0,\mathbf{y})$$

$$-\varepsilon\mathbf{D}(\mathbf{x}/\varepsilon)\cdot\nabla_x R(0,\mathbf{y}) + O(\varepsilon^2) = O(\varepsilon^2) \text{ for } \mathbf{x}\in\partial\omega_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon, \quad (3.10)$$

where $\mathbf{D}(\mathbf{x}/\varepsilon) = \varepsilon^{-1}\mathbf{x}$ for $\mathbf{x}\in\omega_\varepsilon$, and formula (3.5) was used to state that $R(0,\mathbf{y})$ is independent of $\mathbf{y}$. In turn, the second boundary condition (3.3), together with formula (3.1), yields

$$\begin{aligned}\frac{\partial r_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) &= \frac{\partial}{\partial n_x}\Big(h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - \frac{1}{2\pi}\log|\mathbf{x}|^{-1}\Big) \\ &-\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\frac{\partial}{\partial n_x}\Big(\nabla_y R(\mathbf{x},0)\Big) + O(\varepsilon^2)\end{aligned}$$

$$= -\varepsilon \sum_{j=1}^{2} D_j(\varepsilon^{-1}\mathbf{y}) \frac{\partial}{\partial n_x}\Big(\frac{x_j}{2\pi|\mathbf{x}|^2}\Big)$$

$$-\varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\partial}{\partial n_x}\Big(\nabla_y R(\mathbf{x}, 0)\Big) + O(\varepsilon^2) = O(\varepsilon^2), \qquad (3.11)$$

for $\mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon$.

It can also be verified that

$$\int\limits_{\partial\Omega} \frac{\partial}{\partial n_x} r_\varepsilon(\mathbf{x}, \mathbf{y}) dS_x = 0.$$

Indeed,

$$-\int\limits_{\partial\Omega} \frac{\partial}{\partial n_x} r_\varepsilon(\mathbf{x}, \mathbf{y}) dS_x = \int\limits_{\partial\Omega} \frac{\partial}{\partial n_x}\Big\{\mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + \frac{1}{2\pi}\log\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|}$$

$$+ \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y})\Big\} dS_x$$

$$= \varepsilon \int\limits_{\partial\Omega} \frac{\partial}{\partial n_x}\Big\{\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y\Big((2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x}, \mathbf{y})\Big)\Big|_{\mathbf{y}=0}\Big\} dS_x$$

$$= \frac{\varepsilon}{2\pi} \int\limits_{\partial\Omega} \frac{\partial}{\partial n_x}\Big\{\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\mathbf{x}}{|\mathbf{x}|^2}\Big\} dS_x = 0.$$

Using (3.10), (3.11), together with Lemma 3.1, we complete the proof. □

## *3.5 Simpler asymptotic representation of Green's function* $G_\varepsilon^{(D)}$

Two corollaries, which will be formulated here, follow from Theorem 3.1. They include simplified asymptotic formulas for Green's function, which are efficient for the cases where both $\mathbf{x}$ and $\mathbf{y}$ are distant from $F_\varepsilon$ or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$.

**Corollary 3.1.** *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula* (3.1) *is simplified to the form*

$$G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) = N(\mathbf{x}, \mathbf{y}) - (2\pi)^{-1}\log\varepsilon + \zeta_\infty + R(0, 0)$$

$$+ (2\pi)^{-1}\log(|\mathbf{x}||\mathbf{y}|) - \frac{\varepsilon}{2\pi}\mathbf{D}^\infty \cdot (\mathbf{x}|\mathbf{x}|^{-2} + \mathbf{y}|\mathbf{y}|^{-2})$$

$$+ \varepsilon \mathbf{D}^\infty \cdot (\nabla_x R(0,\mathbf{y}) + \nabla_y R(\mathbf{x},0)) + O(\varepsilon^2 |\mathbf{x}|^{-1}|\mathbf{y}|^{-1}), \tag{3.1}$$

*where $R$ is the regular part of the Neumann function $N$ in $\Omega$.*

*Proof.* The estimate (3.1) can be written in the form

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = (2\pi)^{-1} \log(|\boldsymbol{\xi}||\boldsymbol{\eta}|)^{-1} - \zeta_\infty$$
$$+ \frac{\varepsilon}{2\pi} \sum_{j=1}^{2} D_j^\infty \Big( \frac{x_j}{|\mathbf{x}|^2} + \frac{y_j}{|\mathbf{y}|^2} \Big) + O(\varepsilon^2 |\mathbf{x}|^{-1}|\mathbf{y}|^{-1}). \tag{3.2}$$

Using (3.8), (3.1), and (3.2), we obtain

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = & -\frac{1}{2\pi}\log\varepsilon + \frac{1}{2\pi}\log\frac{|\mathbf{x}||\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|} + \zeta_\infty \\ & - \frac{\varepsilon}{2\pi}\sum_{j=1}^{\infty} D_j^\infty \Big( \frac{x_j}{|\mathbf{x}|^2} + \frac{y_j}{|\mathbf{y}|^2} \Big) + O(\varepsilon^2 |\mathbf{x}|^{-1}|\mathbf{y}|^{-1}) \\ & + N(\mathbf{x},\mathbf{y}) - (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0) \\ & + \varepsilon \mathbf{D}^\infty \cdot \Big( \nabla_y R(\mathbf{x},0) + \nabla_x R(0,\mathbf{y}) \Big) \\ & + \varepsilon^2 O(|\mathbf{x}|^{-1} + |\mathbf{y}|^{-1}). \end{aligned} \tag{3.3}$$

Rearranging the terms in (3.3) and taking into account that the remainder terms in the above formula are $O(\varepsilon^2|\mathbf{x}|^{-1}|\mathbf{y}|^{-1})$, we arrive at (3.1). □

Formula (3.1) is efficient when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently distant from $F_\varepsilon$.

The next corollary of Theorem 3.1 gives the representation of $G_\varepsilon^{(D)}$, which is effective for the case where both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$.

**Corollary 3.2.** *The following asymptotic formula for Green's function $G_\varepsilon^{(D)}$ of the boundary value problem* (3.1)–(3.3) *holds*

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = & \,\mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (\mathbf{x} - \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x R(0,\mathbf{y}) \\ & - (\mathbf{y} - \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y R(\mathbf{x},0) \\ & + O(|\mathbf{x}|^2 + |\mathbf{y}|^2 + \varepsilon^2), \end{aligned} \tag{3.4}$$

*for $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (The term $\varepsilon^2$ in the remainder can be omitted if the interior of $F$ is nonempty and contains the origin.)*

*Proof.* Using the Taylor expansion of $R(\mathbf{x},\mathbf{y})$ in a neighborhood of the origin, we reduce the formula (3.1) to the form

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) &= \mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - R(\mathbf{x},\mathbf{y}) + R(0,0) \\ &+ \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + O(\varepsilon^2) \\ &= \mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \qquad (3.5) \\ &- \mathbf{x}\cdot\nabla_x R(0,\mathbf{y}) - \mathbf{y}\cdot\nabla_y R(\mathbf{x},0) + O(|\mathbf{x}|^2+|\mathbf{y}|^2) \\ &+ \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + O(\varepsilon^2). \end{aligned}$$

By rearranging the terms in the above formula, we arrive at (3.4). □

# 4 The Neumann Function for a Planar Domain with a Small Hole or Crack

It is noted in the previous sections that boundary conditions of Dirichlet type were set at a part of the boundary of $\Omega_\varepsilon$. Now, we consider the case where $\partial\Omega_\varepsilon$ is subject to the Neumann boundary conditions. Here, the set $F_\varepsilon$ is the same as in Sect. 2.

The *Neumann function* $N_\varepsilon(\mathbf{x},\mathbf{y})$ for $\Omega_\varepsilon \subset \mathbb{R}^2$ is defined as a solution of the boundary value problem

$$\Delta_x N_\varepsilon(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \qquad (4.1)$$

$$\frac{\partial}{\partial n_x}\Big(N_\varepsilon(\mathbf{x},\mathbf{y}) + (2\pi)^{-1}\log|\mathbf{x}|\Big) = 0, \quad \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon, \qquad (4.2)$$

$$\frac{\partial N_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon. \qquad (4.3)$$

In addition, we require the orthogonality condition, which provides the symmetry of $N_\varepsilon(\mathbf{x},\mathbf{y})$

$$\int_{\partial\Omega} N_\varepsilon(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n}\log|\mathbf{x}|dS_x = 0. \qquad (4.4)$$

The regular part $R_\varepsilon(\mathbf{x},\mathbf{y})$ of the Neumann function is defined by

$$R_\varepsilon(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1} - N_\varepsilon(\mathbf{x},\mathbf{y}).$$

## *4.1 Special solutions of model problems*

As in the previous sections, we consider two limit domains independent of the small parameter $\varepsilon$: the domain $\Omega$ (with no hole), and the unbounded domain

$\mathbb{R}^2 \setminus F$ that represents scaled exterior of the small hole. As always, the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$ will be used.

The Neumann function $N(\mathbf{x}, \mathbf{y})$ of $\Omega$ is defined by (3.1)–(3.3), and the regular part $R(\mathbf{x}, \mathbf{y})$ of $N(\mathbf{x}, \mathbf{y})$ is the same as in (3.4).

We use the vector function $\boldsymbol{\mathcal{D}}$ already defined in Sect. 2.

Another model field to be used is the Neumann function $\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta})$ in $\mathbb{R}^2 \setminus F$, as in (2.12), whose regular part $h_N$ satisfies the problem (2.13)–(2.15).

## *4.2 Maximum modulus estimate for solutions to the Neumann problem in $\Omega_\varepsilon$*

First, we formulate and prove an auxiliary lemma required for the forthcoming estimate of the remainder term in the approximation of $N_\varepsilon$.

**Lemma 4.1.** *Let $u$ be a function in $C(\overline{\Omega}_\varepsilon)$ such that $\nabla u$ is square integrable in a neighborhood $\partial\Omega_\varepsilon$. Also, let $u$ be a solution of the Neumann boundary value problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{4.1}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4.2}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{4.3}$$

*where $\psi \in C(\partial\Omega)$, $\varphi_\varepsilon \in L_\infty(\partial F_\varepsilon)$, and*

$$\int\limits_{\partial F_\varepsilon} \varphi_\varepsilon(\mathbf{x}) ds = 0 \quad \text{and} \quad \int\limits_{\partial\Omega} \psi(\mathbf{x}) ds = 0. \tag{4.4}$$

*We also assume that*

$$\Big| \int\limits_{\partial\Omega} u(\mathbf{x}) \frac{\partial}{\partial n} \big( \log |\mathbf{x}| \big) ds \Big| \leqslant \text{const} \, \{ \|\psi\|_{C(\partial\Omega)} + \varepsilon \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \}. \tag{4.5}$$

*Then there exists a positive constant $C$ independent of $\varepsilon$ and such that*

$$\|u\|_{C(\Omega_\varepsilon)} \leqslant C \big\{ \|\psi\|_{C(\partial\Omega)} + \varepsilon \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \big\}. \tag{4.6}$$

*Proof.* (a) We use the operators $\mathfrak{N}$ and $\mathfrak{N}_\Omega$ of the model problems (2.7)–(2.9) and (3.7)–(3.9) introduced in Sects. 2 and 3.

(b) We begin with the case of the homogeneous boundary condition on $\partial\Omega$, i.e.,

$$\Delta u_1(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{4.7}$$

$$\frac{\partial u_1}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{4.8}$$

$$\frac{\partial u_1}{\partial n}(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{4.9}$$

where the right-hand side $\varphi_\varepsilon$ is such that

$$\int\limits_{\partial F_\varepsilon} \varphi_\varepsilon(\mathbf{x}) ds = 0.$$

The operator $\mathfrak{N}_\varepsilon$ is defined as in (2.11), so that

$$(\mathfrak{N}_\varepsilon \varphi_\varepsilon)(\mathbf{x}) = (\mathfrak{N}\varphi)(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-1}\varphi(\varepsilon^{-1}\mathbf{x})$.

The solution $u_1$ is sought in the form

$$u_1 = \mathfrak{N}_\varepsilon g_\varepsilon - \mathfrak{N}_\Omega\Big(\frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon g_\varepsilon)_{\partial\Omega}\Big), \tag{4.10}$$

where $g_\varepsilon$ is an unknown function such that

$$\int\limits_{\partial F} g(\boldsymbol{\xi}) ds_\xi = 0.$$

By Lemma 2.1, we have

$$|\mathfrak{N}g(\boldsymbol{\xi})| \leqslant C\varepsilon \|g\|_{L_\infty(\partial F)} \tag{4.11}$$

and

$$\max_{\overline{\Omega}_\varepsilon} |\mathfrak{N}_\varepsilon g_\varepsilon| \leqslant C\varepsilon \|g_\varepsilon\|_{L_\infty(\partial F)}. \tag{4.12}$$

From (4.10) it follows that $\frac{\partial}{\partial n} u_1(\mathbf{x}) = 0$ when $\mathbf{x} \in \partial\Omega$, and on the boundary $\partial F_\varepsilon$ we have

$$\varphi_\varepsilon = g_\varepsilon + S_\varepsilon g_\varepsilon, \tag{4.13}$$

where

$$S_\varepsilon g_\varepsilon = -\frac{\partial}{\partial n}\Big(\mathfrak{N}_\Omega\big(\frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon g_\varepsilon)_{\partial\Omega}\big)\Big) \quad \text{on} \quad \partial F_\varepsilon. \tag{4.14}$$

Taking into account Lemma 2.1 and the definitions of $\mathfrak{N}_\Omega$ and $\mathfrak{N}_\varepsilon$, as in (3.6) and (2.6), (2.11), we deduce that

$$\max_{\partial\Omega} |\nabla(\mathfrak{N}_\varepsilon g_\varepsilon)| \leqslant \text{const } \varepsilon^2 \|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}$$

and

$$\|S_\varepsilon g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \leqslant \text{const} \;\; \varepsilon^2 \|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}.$$

Owing to the smallness of the norm of the operator $S_\varepsilon$ we can write

$$\|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \leqslant \text{const} \;\; \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}.$$

Following (3.10), (3.11), (4.10), and (4.12), we deduce (4.5) and

$$\max_{\overline{\Omega}_\varepsilon} |u_1| \leqslant \text{const} \;\; \varepsilon \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{4.15}$$

(c) Next, we consider the problem (4.1)–(4.4) with the homogeneous data on $\partial\omega_\varepsilon$. The corresponding solution $u_2$ is written in the form

$$u_2 = \mathfrak{N}_\Omega \psi + v, \tag{4.16}$$

where the harmonic function $v$ satisfies zero boundary condition on $\partial\Omega$, whereas the condition (4.9) is replaced by

$$\frac{\partial}{\partial n} v(\mathbf{x}) = -\frac{\partial}{\partial n}\big(\mathfrak{N}_\Omega \psi\big)(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon,$$

and, by part (b),

$$\max_{\overline{\Omega}_\varepsilon} |v| \leqslant \text{const} \;\; \|\psi\|_{C(\partial\Omega)}.$$

The function $v$ and hence $u_2$ satisfies (4.5).

Following (3.10), (3.11), and (4.16), we deduce

$$\max_{\overline{\Omega}_\varepsilon} |u_2| \leqslant \text{const} \;\; \|\psi\|_{C(\partial\Omega)}. \tag{4.17}$$

Combining the estimates (4.15) and (4.17), we complete the proof. □

### 4.3 Asymptotic approximation of $N_\varepsilon$

Now, we state the theorem, which gives a uniform asymptotic formula for the Neumann function $N_\varepsilon$.

**Theorem 4.1.** *The Neumann function $N_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\eta})$ of the domain $\Omega_\varepsilon$ defined in* (4.1)–(4.4) *satisfies*

$$\begin{aligned} N_\varepsilon(\mathbf{x}, \mathbf{y}) = \;& N(\mathbf{x}, \mathbf{y}) - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ & + \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y}) \\ & + \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + r_\varepsilon(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{4.1}$$

*where*

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \leqslant \text{Const}\ \ \varepsilon^2 \tag{4.2}$$

*uniformly with respect to* $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

*Proof.* We begin with a formal argument leading to the approximation (4.1). Consider the first three terms on the right-hand side of (4.1). Let

$$r_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = N_\varepsilon(\mathbf{x},\mathbf{y}) - N(\mathbf{x},\mathbf{y}) + h_N(\boldsymbol{\xi},\boldsymbol{\eta}) - \varepsilon\boldsymbol{\mathcal{D}}(\boldsymbol{\xi})\cdot\nabla_x R(0,\mathbf{y}). \tag{4.3}$$

The function $r_\varepsilon^{(1)}$ is harmonic in $\Omega_\varepsilon$, and the direct substitution into the boundary conditions (4.2) and (4.3) gives

$$\begin{aligned}\frac{\partial r_\varepsilon^{(1)}}{\partial n_x}(\mathbf{x},\mathbf{y}) = &-\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1}\Big) + \frac{\partial}{\partial n_x}\big(h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\big)\\ &+\mathbf{n}\cdot\nabla_x R(0,\mathbf{y}) - \varepsilon\frac{\partial}{\partial n_x}\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + O(\varepsilon)\\ &= O(\varepsilon)\ \text{ for } \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{\partial r_\varepsilon^{(1)}}{\partial n_x}(\mathbf{x},\mathbf{y}) &= \frac{\partial}{\partial n_x}\big(h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\big) + O(\varepsilon^2)\\ &= \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\frac{\partial}{\partial n_x}\nabla_y R(\mathbf{x},0) + O(\varepsilon^2)\\ &\quad \text{for } \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon.\end{aligned} \tag{4.5}$$

Thus, $r_\varepsilon^{(1)}$ can be approximated as

$$r_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + O(\varepsilon^2),$$

and, together with the representation (4.3), this leads to the required formula (4.1).

Finally, the direct substitution of (4.1) into (4.1)–(4.3) yields that the remainder term $r_\varepsilon(\mathbf{x},\mathbf{y})$ satisfies the problem (4.1)–(4.4), with

$$\max_{\mathbf{x}\in\partial\Omega}|\psi(\mathbf{x},\mathbf{y})| \leqslant \text{Const}\ \ \varepsilon^2$$

and

$$\max_{\mathbf{x}\in\partial F_\varepsilon}|\varphi_\varepsilon(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})| \leqslant \text{Const}\ \ \varepsilon$$

for all $\mathbf{y}\in\Omega_\varepsilon$. Then the estimate (4.2) follows from Lemma 4.1. □

## 4.4 Simpler asymptotic representation of the Neumann function $N_\varepsilon$

Two corollaries, formulated in this section, follow from Theorem 4.1. They include asymptotic formulas for the Neumann function, which are efficient when either both $\mathbf{x}$ and $\mathbf{y}$ are distant from $F_\varepsilon$ or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$.

**Corollary 4.1.** *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then*

$$\begin{aligned} N_\varepsilon(\mathbf{x},\mathbf{y}) = N(\mathbf{x},\mathbf{y}) &- \frac{\varepsilon^2}{4\pi^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}^T}{|\mathbf{y}|^2} \\ &+ \frac{\varepsilon^2}{2\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\nabla_x R(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^2}\boldsymbol{\mathcal{P}}\nabla_y R(\mathbf{x},0)\Big\} \\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2}), \end{aligned} \tag{4.1}$$

*where* $R$ *is the regular part of the Neumann function* $N$ *in* $\Omega$, *and* $\boldsymbol{\mathcal{P}}$ *is the dipole matrix for* $F$, *as defined in* (2.1).

*Proof.* The proof is similar to that of Corollary 2.1, and it uses formula (2.9) for the regular part $h_N$ of the Neumann function in $\mathbb{R}^2 \setminus F$, together with the asymptotic representation (2.1) of the dipole fields $\mathcal{D}_j$ in $\mathbb{R}^2 \setminus F$. □

Next, we state a proposition similar to Corollaries 2.2 and 3.2 formulated earlier for Green's functions $G_\varepsilon^{(D)}$ and $G_\varepsilon^{(N)}$.

**Corollary 4.2.** *The Neumann function* $N_\varepsilon$, *defined by* (4.1)–(4.4), *satisfies the asymptotic formula*

$$\begin{aligned} N_\varepsilon(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - R(0,0) - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ - (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x R(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y R(\mathbf{x},0) \\ + O(|\mathbf{x}|^2 + |\mathbf{y}|^2 + \varepsilon^2), \end{aligned} \tag{4.2}$$

*for* $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (*As in Corollaries* 2 *and* 4, $\varepsilon^2$ *in the remainder can be omitted if the interior of* $F$ *is nonempty and contains the origin.*)

*Proof.* The proof is similar to that of Corollary 2.2 and employs the linear approximation of the regular part $R$ of the Neumann function in a neighborhood of the origin. □

Although the formulation of Corollary 4.2 is valid for all $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$, the asymptotic formula (4.2) becomes effective when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$.

# 5 Asymptotic approximations of Green's kernels for mixed and Neumann problems in three dimensions

This section includes asymptotic formulas for Green's kernels $G_\varepsilon^{(D)}, G_\varepsilon^{(N)}$ and $N_\varepsilon$ in $\Omega_\varepsilon \subset \mathbb{R}^3$. The special solutions of model problems differ from the corresponding solutions used for the two-dimensional case. The uniform asymptotic formulas of Green's kernels are accompanied by simpler representations, which are efficient when certain constraints are imposed on the independent variables. The proofs, which do not require new ideas compared with the two-dimensional case, are omitted.

## 5.1 *Special solutions of model problems in limit domains*

Here, we describe the functions $G$, $\mathcal{G}$, $N$, and $\mathcal{N}$ defined in the limit domains and used for the approximation of Green's kernels.

1. The notation $G$ is used for Green's function of the Dirichlet problem in $\Omega \subset \mathbb{R}^3$:

$$G(\mathbf{x}, \mathbf{y}) = (4\pi|\mathbf{x} - \mathbf{y}|)^{-1} - H(\mathbf{x}, \mathbf{y}). \tag{5.1}$$

Here, $H$ is the regular part of $G$, and it is a unique solution of the Dirichlet problem

$$\Delta_x H(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega, \tag{5.2}$$

$$H(\mathbf{x}, \mathbf{y}) = (4\pi|\mathbf{x} - \mathbf{y}|)^{-1}, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega. \tag{5.3}$$

2. Green's function $\mathcal{G}$ for the Dirichlet problem in $\mathbb{R}^3 \setminus F$ is defined as a unique solution of the problem

$$\Delta_\xi \mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \delta(\boldsymbol{\xi} - \boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{5.4}$$

$$\mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi} \in \partial F, \ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{5.5}$$

$$\mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta}) \to 0 \ \text{ as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F. \tag{5.6}$$

Here, $F$ is a compact set of positive harmonic capacity.

The regular part $h$ of Green's function $\mathcal{G}$ is

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = (4\pi|\boldsymbol{\xi} - \boldsymbol{\eta}|)^{-1} - \mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta}). \tag{5.7}$$

3. The components of the vector field $\mathbf{D}(\boldsymbol{\xi}) = (D_1(\boldsymbol{\xi}), D_2(\boldsymbol{\xi}), D_3(\boldsymbol{\xi}))$ (compare with (3.9)–(3.11)), for $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus F$, satisfy the problem

$$\Delta D_j(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^3 \setminus F, \tag{5.8}$$

$$D_j(\boldsymbol{\xi}) = \xi_j, \quad \boldsymbol{\xi} \in \partial F, \tag{5.9}$$

$$D_j(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty. \tag{5.10}$$

We use the matrix $\boldsymbol{\mathcal{T}} = (\mathcal{T}_{jk})_{j,k=1}^3$ of coefficients in the asymptotic representation of $D_j$ at infinity

$$D_j(\boldsymbol{\xi}) = \frac{1}{4\pi} \sum_{k=1}^{3} \frac{\mathcal{T}_{jk}\xi_k}{|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}). \tag{5.11}$$

The symmetry of $\boldsymbol{\mathcal{T}}$ is verified by applying Green's formula in $B_R \setminus F$ to $\xi_j - D_j(\boldsymbol{\xi})$ and $D_k(\boldsymbol{\xi})$ and taking the limit $R \to \infty$. We have

$$\int_{\partial B_R} \Big\{(\xi_j - D_j(\boldsymbol{\xi}))\frac{\partial D_k(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi})\Big(\frac{\xi_j}{|\boldsymbol{\xi}|} - \frac{\partial D_j(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|}\Big)\Big\}dS$$
$$+ \int_{\partial F} D_k(\boldsymbol{\xi})\Big(\frac{\partial D_j(\boldsymbol{\xi})}{\partial n} - n_j\Big)dS = 0, \tag{5.12}$$

where $\partial/\partial n$ is the normal derivative in the direction of the interior normal with respect to $F$. As $R \to \infty$, the first integral $\mathcal{I}(\partial B_R)$ on the left-hand side of (5.12) gives

$$\lim_{R\to\infty} \mathcal{I}(\partial B_R) = \lim_{R\to\infty} \int_{\partial B_R} \Big\{\xi_j \frac{\partial D_k(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi})\frac{\xi_j}{|\boldsymbol{\xi}|}\Big\}dS$$
$$= -\frac{3}{4\pi} \int_{\partial B_1} \sum_{q=1}^{3} \mathcal{T}_{kq}\xi_q\xi_j dS = -\mathcal{T}_{kj}. \tag{5.13}$$

The second integral $\mathcal{I}(\partial F)$ on the left-hand side of (5.12) becomes

$$\mathcal{I}(\partial F) = -\int_{\partial F} \xi_k n_j dS + \int_{\partial F} D_k(\boldsymbol{\xi})\frac{\partial D_j(\boldsymbol{\xi})}{\partial n} dS$$
$$= \delta_{jk} \ \mathrm{meas}_3(F) + \int_{\mathbb{R}^3\setminus F} \nabla D_k(\boldsymbol{\xi}) \cdot \nabla D_j(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{5.14}$$

where $\mathrm{meas}_3(F)$ is the three-dimensional Lebesgue measure of $F$. Using (5.13) and (5.14), we deduce

$$\mathcal{T}_{kj} = \delta_{jk} \ \mathrm{meas}_3(F) + \int_{\mathbb{R}^3\setminus F} \nabla D_k(\boldsymbol{\xi}) \cdot \nabla D_j(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{5.15}$$

which implies that $\boldsymbol{\mathcal{T}}$ is *symmetric and positive definite.*

4. The Neumann function $N(\mathbf{x}, \mathbf{y})$ in $\Omega \subset \mathbb{R}^3$ and its regular part are defined as follows:

$$\Delta N(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega \subset \mathbb{R}^3, \tag{5.16}$$

$$\frac{\partial}{\partial n_x}\Big(N(\mathbf{x}, \mathbf{y}) - (4\pi)^{-1}|\mathbf{x}|^{-1}\Big) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega, \tag{5.17}$$

and

$$\int_{\partial\Omega} N(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial n_x} |\mathbf{x}|^{-1} ds_x = 0, \tag{5.18}$$

where the last condition (3.3) implies the symmetry of $N(\mathbf{x}, \mathbf{y})$. The regular part of the Neumann function in three dimensions is defined by

$$R(\mathbf{x}, \mathbf{y}) = (4\pi)^{-1}|\mathbf{x} - \mathbf{y}|^{-1} - N(\mathbf{x}, \mathbf{y}). \tag{5.19}$$

5. In this section, the notation $\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta})$ is used for the Neumann function in $\mathbb{R}^3 \setminus F$, where $F$ is a compact closure of a domain with a smooth boundary, and $\mathcal{N}$ is defined by

$$\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta}) = (4\pi)^{-1}|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} - h_N(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{5.20}$$

where $h_N$ is the regular part of $\mathcal{N}$ subject to

$$\Delta_\xi h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{5.21}$$

$$\frac{\partial h_N}{\partial n_\xi}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{4\pi}\frac{\partial}{\partial n_\xi}(|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}), \quad \boldsymbol{\xi} \in \partial F, \ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{5.22}$$

$$h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) \to 0 \ \text{ as } |\boldsymbol{\xi}| \to \infty, \ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F. \tag{5.23}$$

The smoothness assumption on $\partial F$ here and in the sequel is introduced for the simplicity of proofs and can be considerably weakened. In particular, the case of a piece-wise smooth planar crack can be included.

We note that the Neumann function $\mathcal{N}$ just defined is symmetric, i.e., $\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathcal{N}(\boldsymbol{\eta}, \boldsymbol{\xi})$.

6. The definition of the dipole vector field $\boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = (\mathcal{D}_1(\boldsymbol{\xi}), \mathcal{D}_2(\boldsymbol{\xi}), \mathcal{D}_3(\boldsymbol{\xi}))$ is similar to (2.16)–(2.18) with $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus F$. The components of the three-dimensional dipole matrix $\boldsymbol{\mathcal{P}} = (\mathcal{P}_{jk})_{j,k=1}^3$ appear in the asymptotic representation of $\mathcal{D}_j(\boldsymbol{\xi})$ at infinity

$$\mathcal{D}_j(\boldsymbol{\xi}) = \frac{1}{4\pi}\sum_{k=1}^{3}\frac{\mathcal{P}_{jk}\xi_k}{|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}). \tag{5.24}$$

Similar to Sect. 2.2, it can be proved that the *dipole matrix* $\boldsymbol{\mathcal{P}}$ *for the hole* $F$ *is symmetric and negative definite.*

## *5.2 Approximations of Green's kernels*

The following assertions hold for uniform asymptotic approximations in three-dimensional domains with small holes (or cracks) or inclusions.

**Theorem 5.1.** *Green's function* $G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y})$ *for the mixed problem with the Neumann data on* $\partial F_\varepsilon$ *and the Dirichlet data on* $\partial\Omega$ *has the asymptotic representation*

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}\mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (4\pi)^{-1}|\mathbf{x}-\mathbf{y}|^{-1}$$

$$+\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0) + r_\varepsilon(\mathbf{x},\mathbf{y}), \tag{5.1}$$

*where* $\boldsymbol{\mathcal{D}}$ *is the three-dimensional dipole vector function in* $\mathbb{R}^3\setminus F$*, and* $\mathcal{N}$ *is the Neumann function in* $\mathbb{R}^3\setminus F$*, vanishing at infinity. Here,*

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \leqslant \text{Const } \varepsilon^2 \tag{5.2}$$

*uniformly with respect to* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$*.*

The proof follows the same algorithm as in Theorem 2.1.

Now, we give analogues of Corollaries 2.1 and 2.2 formulated earlier in Sect. 2.7.

**Corollary 5.1.** *Let* $\min\{|\mathbf{x}|,|\mathbf{y}|\} > 2\varepsilon$*. Then the asymptotic formula* (5.1) *is simplified to the form*

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) & \\ &+\frac{\varepsilon^3}{4\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\nabla_x H(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3}\boldsymbol{\mathcal{P}}\nabla_y H(\mathbf{x},0)\Big\} \\ &-\frac{\varepsilon^3}{(4\pi)^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}}{|\mathbf{y}|^3} \\ &+O(\varepsilon^2+\varepsilon^4(|\mathbf{x}|+|\mathbf{y}|)|\mathbf{x}|^{-3}|\mathbf{y}|^{-3}), \end{aligned} \tag{5.3}$$

*where* $H$ *is the regular part of Green's function* $G$ *in* $\Omega$*, and* $\boldsymbol{\mathcal{P}}$ *is the dipole matrix for* $F$*, as defined in* (5.24)*.*

The next assertion is similar to Corollary 2.2 of Sect. 2.7.

**Corollary 5.2.** *The following asymptotic formula for Green's function* $G_\varepsilon^{(N)}$ *holds*

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = \varepsilon^{-1}\mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(0,0) \\ - (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x H(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y H(\mathbf{x},0) \\ + O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2) \quad (5.4)$$

*for* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$. *(As in Corollary 2.2,* $\varepsilon^2$ *in the remainder can be omitted if the interior of* $F$ *is nonempty and contains the origin.)*

In turn, for the case where the Neumann and Dirichlet boundary conditions are set on $\partial\Omega$ and $\partial F_\varepsilon$ respectively, the modified version of formula (3.1) is given by

**Theorem 5.2.** *The Green's function* $G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y})$ *for the mixed problem with the Dirichlet data on* $\partial F_\varepsilon$ *and the Neumann data on* $\partial\Omega$*, admits the asymptotic representation*

$$G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = \varepsilon^{-1}\mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + N(\mathbf{x},\mathbf{y}) - (4\pi)^{-1}|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0) \\ +\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + r_\varepsilon(\mathbf{x},\mathbf{y}), \quad (5.5)$$

*where*

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \leqslant \text{Const}\ \varepsilon^2,$$

*which is uniform with respect to* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$.

The proof is similar to that of Theorem 3.1. We note that unlike the two-dimensional case, in three dimensions no orthogonality condition is required to ensure the decay of the solution of the exterior Dirichlet problem in $\mathbb{R}^3\setminus F$.

The analogues of Corollaries 3.1 and 3.2 are formulated as follows.

**Corollary 5.3.** *Let* $\min\{|\mathbf{x}|,|\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula* (5.5) *is simplified to the form*

$$G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = N(\mathbf{x},\mathbf{y}) + R(0,0) \\ + \frac{\varepsilon^3}{4\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{T}}\nabla_x R(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3}\boldsymbol{\mathcal{T}}\nabla_y R(\mathbf{x},0)\Big\} \\ - \frac{\varepsilon^3}{(4\pi)^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{T}}\frac{\mathbf{y}}{|\mathbf{y}|^3} \\ + O(\varepsilon^2 + \varepsilon^4(|\mathbf{x}|+|\mathbf{y}|)|\mathbf{x}|^{-3}|\mathbf{y}|^{-3}), \quad (5.6)$$

*where* $R$ *is the regular part of the Neumann function* $N$ *in* $\Omega$*, and* $\boldsymbol{\mathcal{T}}$ *is the matrix of coefficients in* (5.11).

The next assertion is similar to Corollary 3.2 of Sect. 3.5.

**Corollary 5.4.** *The following asymptotic formula for Green's function* $G_\varepsilon^{(D)}$ *holds*

$$
\begin{aligned}
G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}\mathcal{G}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
&\quad - (\mathbf{x}-\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x R(0,\mathbf{y}) - (\mathbf{y}-\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y R(\mathbf{x},0) \\
&\quad + O(\varepsilon^2+|\mathbf{x}|^2+|\mathbf{y}|^2) \qquad (5.7)
\end{aligned}
$$

*for* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$. *(The term* $\varepsilon^2$ *in the remainder can be omitted if the interior of* $F$ *is nonempty and contains the origin.)*

Finally, we consider the *Neumann function* $N_\varepsilon(\mathbf{x},\mathbf{y})$ for $\Omega_\varepsilon\subset\mathbb{R}^3$. Here, $\Omega_\varepsilon=\Omega\setminus F_\varepsilon$, and $F_\varepsilon$ is the small hole with a smooth boundary. We define $N_\varepsilon$ as a solution of the following boundary value problem

$$\Delta_x N_\varepsilon(\mathbf{x},\mathbf{y})+\delta(\mathbf{x}-\mathbf{y})=0,\quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \qquad (5.8)$$

$$\frac{\partial}{\partial n_x}\Big(N_\varepsilon(\mathbf{x},\mathbf{y})-(4\pi)^{-1}|\mathbf{x}|^{-1}\Big)=0,\quad \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon, \qquad (5.9)$$

$$\frac{\partial N_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y})=0,\quad \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon. \qquad (5.10)$$

In addition, we require the orthogonality condition, which provides the symmetry of $N_\varepsilon(\mathbf{x},\mathbf{y})$

$$\int_{\partial\Omega} N_\varepsilon(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n}|\mathbf{x}|^{-1}dS_x=0. \qquad (5.11)$$

The asymptotic approximation of $N_\varepsilon$ is given by

**Theorem 5.3.** *The Neumann function* $N_\varepsilon(\boldsymbol{\xi},\boldsymbol{\eta})$ *for the domain* $\Omega_\varepsilon$*, defined in* (5.8)–(5.11) *satisfies the asymptotic formula*

$$
\begin{aligned}
N_\varepsilon(\mathbf{x},\mathbf{y}) = {} & N(\mathbf{x},\mathbf{y})-\varepsilon^{-1}h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})+\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) \\
& +\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0)+r_\varepsilon(\mathbf{x},\mathbf{y}), \qquad (5.12)
\end{aligned}
$$

*where*

$$|r_\varepsilon(\mathbf{x},\mathbf{y})|\leqslant \text{Const }\ \varepsilon^2 \qquad (5.13)$$

*uniformly with respect to* $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$. *Here,* $\boldsymbol{\mathcal{D}}$ *is the three-dimensional dipole vector function in* $\mathbb{R}^3\setminus F$*, and* $h_N$ *is the regular part of the Neumann function* $\mathcal{N}$ *in* $\mathbb{R}^3\setminus F$*, vanishing at infinity. The Neumann function* $N$ *in* $\Omega$ *and its regular part* $R$ *are the same as in* (5.16)–(5.19)*.*

The proof follows the same algorithm as in Theorem 4.1.

At last, we formulate analogues of Corollaries 4.1 and 4.2 for the Neumann problem in $\Omega_\varepsilon$.

**Corollary 5.5.** *Let* $\min\{|\mathbf{x}|,|\mathbf{y}|\}>2\varepsilon$. *Then* $N_\varepsilon(\mathbf{x},\mathbf{y})$ *is approximated in the form*

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = N(\mathbf{x},\mathbf{y}) - \frac{\varepsilon^3}{(4\pi)^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}^T}{|\mathbf{y}|^3}$$
$$+ \frac{\varepsilon^3}{4\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\nabla_x R(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3}\boldsymbol{\mathcal{P}}\nabla_y R(\mathbf{x},0)\Big\} \quad (5.14)$$
$$+ O(\varepsilon^2 + \varepsilon^4(|\mathbf{x}|+|\mathbf{y}|)|\mathbf{x}|^{-3}|\mathbf{y}|^{-3}),$$

*where $R$ is the regular part of the Neumann function in $\Omega$, and $\boldsymbol{\mathcal{P}}$ is the dipole matrix for $F$, as defined in* (5.24).

When both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to $F_\varepsilon$, the asymptotic approximation of $N_\varepsilon$ is given in the next assertion.

**Corollary 5.6.** *The Neumann function $N_\varepsilon$ satisfies the asymptotic formula*

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = \varepsilon^{-1}\mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - R(0,0)$$
$$- (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x R(0,\mathbf{y})$$
$$- (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y R(\mathbf{x},0)$$
$$+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2) \quad (5.15)$$

*for $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$. The term $\varepsilon^2$ in the remainder can be omitted if the interior of $F$ is nonempty and contains the origin.*

**Acknowledgement.** This work was supported by the UK Engineering and Physical Sciences Research Council via the research (grant no. EP/F005563/1).

We would like to thank Michael Nieves (University of Liverpool) for providing a numerical example mentioned in Introduction.

## References

1. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin-Heidelberg-New York (1983)
2. Hadamard, J.: Sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. Mémoire couronné en 1907 par l'Académie des Sciences **33**, no 4, 515-629 (1907)
3. Landkof, N.S.: Foundations of Modern Potential Theory. Spinger, Berlin-Heidelberg-New York (1972)
4. Maz'ya, V.: Sobolev Spaces. Spinger, Berlin-Heidelberg-New York (1985
5. Maz'ya, V., Movchan, A.: Uniform asymptotic formulae for Green's kernels in regularly and singularly perturbed domains. C. R. Acad. Sci. Paris, Ser. Math. **343**, 185-190 (2006)
6. Maz'ya, V., Movchan, A.: Uniform asymptotic formulae for Green's functions in singularly perturbed domains. J. Comp. Appl. Math.. **228**, 194-206 (2007)

7. Maz'ya, V.G., Plamenevskii, B.A.: Estimates in $L_p$ and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. Transl., Ser. 2, Am. Math. Soc. **123**, 1-56 (1984)
8. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Princeton (1951)

# Finsler Structures and Wave Propagation

Michael Taylor

*Dedicated to the memory of*
*the Great Analyst S.L. Sobolev*

**Abstract** We discuss connections between the study of wave propagation for general classes of hyperbolic PDEs (beyond the "standard wave equation") and aspects of Finsler geometry. In particular, we investigate how understanding of the behavior of differential operators (and pseudodifferential operators) arising in such study can enhance one's understanding of Finsler geometry. We also discuss a problem in harmonic analysis motivated by a construction of Katok in Finsler geometry, which gives rise to an interesting variant of the Pinsky phenomenon, for pointwise Fourier inversion.

## 1 Introduction

A Finsler metric on a smooth manifold $M$ is a $C^\infty$ function

$$F : TM \setminus 0 \longrightarrow (0, \infty) \tag{1.1}$$

with the following properties:

$$F(x, \lambda v) = \lambda F(x, v) \quad \forall\ v \in T_x M \setminus 0,\ \lambda \in (0, \infty), \tag{1.2}$$

and, if we set

$$f(x, v) = \frac{1}{2} F(x, v)^2, \tag{1.3}$$

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V. Isakov (ed.), *Sobolev Spaces in Mathematics III, International Mathematical Series*. 317
doi: 10.1007/978-0-387-85652-0, 

then

$$D_v^2 f(x,v) \text{ is positive definite,} \tag{1.4}$$

as a symmetric bilinear form on $T_xM$, for each $x \in M$ and each nonzero $v \in T_xM$. The hypotheses (1.2)–(1.4) say that $F(x,\cdot)$ defines a "Minkowski norm" on $T_xM$ for each $x \in M$.

An example of a Finsler metric is $F(x,v) = g_x(v,v)^{1/2}$, where $g$ is a Riemannian metric tensor on $M$. Many objects studied in Riemannian geometry are of interest in the broader context of Finsler geometry. For example, one looks for "length minimizing" curves $\gamma : [a,b] \to M$, extrema for

$$L(\gamma) = \int_a^b F(\gamma(t), \gamma'(t))\, dt, \tag{1.5}$$

with fixed endpoints. As is the case for Riemannian geometry, $L(\gamma)$ is invariant under reparametrization of $\gamma$; one normalizes by demanding that $\gamma$ have constant speed, i.e., $F(\gamma(t), \gamma'(t)) = c$. Furthermore, as in the Riemannian case, it is effective to look at the problem of extremizing

$$E(\gamma) = \int_a^b f(\gamma(t), \gamma'(t))\, dt. \tag{1.6}$$

There is a standard technique for converting the Euler–Lagrange equation for extrema of (1.6) to Hamiltonian form, via a Legendre transform

$$\varXi : TM \setminus 0 \longrightarrow T^*M \setminus 0. \tag{1.7}$$

The hypothesis (1.4) serves to guarantee that $\varXi$ is a diffeomorphism (extending to a bi-Lipschitz map $\varXi : TM \to T^*M$). The geodesic flow on $TM \setminus 0$ is carried by $\varXi$ to the flow on $T^*M \setminus 0$ generated by the Hamiltonian field $H_\varPhi$, or $H_\varphi = \varPhi H_\varPhi$, where

$$\varPhi = F \circ \varXi^{-1}, \quad \varphi = \frac{1}{2}\varPhi^2 = f \circ \varXi^{-1}. \tag{1.8}$$

Details on this are given in Sect. 2.

It is shown that $\varPhi(x,\xi)$ satisfies analogues of 1.2–1.4, i.e., $\varPhi(x,\cdot)$ defines a Minkowski norm on $T_x^*M$ for each $x \in M$. We call such $\varPhi(x,\xi)$ a *Finsler symbol*. Calculations done in Sect. 2 show that the correspondence $F \mapsto \varPhi$ is reversible; not only does each Finsler metric determine a Finsler symbol, but also each Finsler symbol determines a Finsler metric.

Thus, there are two approaches to defining a Finsler structure. One is to specify directly a Finsler metric $F(x,v)$. The other is to specify directly a Finsler symbol $\varPhi(x,\xi)$. The correspondence

$$F(x, v) \leftrightarrow \Phi(x, \xi) \tag{1.9}$$

has been the object of some study. R. Miron and colleagues call $M$ equipped with a Finsler symbol $\Phi(x, \xi)$ on $T^*M$ a "Cartan space," and the correspondence (1.9) the "Finsler–Cartan $\mathcal{L}$-duality." See [6], particularly Chapt. 7, for a discussion. This correspondence is also very much in evidence in Ziller's presentation [13] of a construction of Katok [5].

One purpose of this note is to point out how the approach of specifying and studying a Finsler symbol arises naturally in the analysis of problems in wave propagation. The connection between Finsler symbols and wave equations is readily explained. Finsler symbols are principal symbols of first order, elliptic, self-adjoint pseudodifferential operators, and such pseudodifferential operators arise in the analysis of various hyperbolic PDEs. For the standard wave equation for $u = u(t, x)$ defined on $\mathbb{R} \times M$:

$$u_{tt} - \Delta u = 0, \tag{1.10}$$

where $\Delta$ is the Laplace–Beltrami operator defined by a Riemann metric tensor on $M$, the solution is given in terms of $e^{\pm it\sqrt{-\Delta}}$, and the symbol of $\sqrt{-\Delta}$ is $\Phi(x, \xi)$, the length of $\xi \in T_x^*M$ given by the inner product on $T_x^*$ induced by the Riemann metric tensor. This is all within the Riemannian geometry setting. However, other hyperbolic PDEs give rise to first order pseudodifferential operators whose symbols are frequently other Finsler symbols (see Sect. 3 for further development of this theme).

In Sect. 4, we recall some Finsler metrics on spheres, produced by Katok [5] to have geodesic flows with notably few closed orbits. We embark on our second main goal of this note, which is to study the spectral behavior of the pseudodifferential operators associated with these Finsler metrics. We consider eigenfunction expansions of certain piecewise smooth functions, in terms of the eigenfunctions of these operators, and examine their pointwise behavior. We find that interesting variants of the Pinsky phenomenon arise, producing in some cases infinite sets of points at which such an eigenfunction expansion has an oscillatory divergence.

Sobolev introduced methods of functional analysis that transformed the theory of partial differential equations. He was particularly instrumental in applying his new ideas to the theory of linear hyperbolic equations. These ideas have a strong influence on the analytical techniques of this paper.

## 2 Finsler Metrics and Finsler Symbols

As stated in Sect. 1, here we work out the application of the Legendre transform to converting the problem of extremizing (1.6) to a Hamiltonian differential equation. The standard recipe (see [10, Chapt. I, Sect. 12]) for defining the Legendre transform $\Xi$ in (1.3) is $\Xi(x,v)=(x,\xi)$, with

$$\xi = D_v f(x,v). \tag{2.1}$$

Then the Euler–Lagrange equation for $(x,v)$ is converted to the Hamiltonian equation

$$(x',\xi') = (D_\xi\varphi, -D_x\varphi) = H_\varphi, \tag{2.2}$$

where

$$\varphi(x,\xi) = D_v f(x,v)v - f(x,v) = f(x,v). \tag{2.3}$$

The first identity in (2.3) is the general prescription (see [10, Chapt. I, (12.17)]) and the second is a consequence of Euler's identity $D_v f(x,v)v = 2f(x,v)$, valid when $f(x,v)$ is positively homogeneous of degree 2 in $v$. Note that since $\varphi$ is constant on each integral curve of $H_\varphi$, it follows that $f(\gamma(t),\gamma'(t))$ is constant on the image $(\gamma(t),\gamma'(t))$ under $\Xi^{-1}$ of such a integral curve. Such a curve is then a constant-speed extremum for (1.5).

It is easily verified that the hypothesis (1.4) of strong convexity implies that (2.1) provides a diffeomorphism $\Xi : TM\backslash 0 \to T^*M\backslash 0$. Note in particular that

$$D_v\xi(x,v) = D_v^2 f(x,v) \in T_x^* \otimes T_x^* \approx \mathrm{Hom}(T_x, T_x^*). \tag{2.4}$$

We also note that

$$v(x,\xi) = D_\xi\varphi(x,\xi). \tag{2.5}$$

This follows from the first part of (2.2) since also $v = x'$. We can also deduce (2.5) directly from (2.3) and (2.4), as follows. First, differentiate $\varphi(x,\xi) = f(x,v)$, obtaining

$$D_\xi\varphi(x,\xi)\ D_v\xi(x,v) = D_v f(x,v). \tag{2.6}$$

Then plug in (2.4), to get

$$D_\xi\varphi(x,\xi)\ D_v^2 f(x,v) = D_v f(x,v). \tag{2.7}$$

On the other hand, Euler's identity gives

$$v\, D_v^2 f(x,v) = D_v f(x,v). \tag{2.8}$$

Comparing the left-hand sides of (2.7) and (2.8) and noting that $D_v^2 f(x,v)$ is invertible, by (1.4), we have the asserted identity (2.5).

Having the function $\varphi(x,\xi)$ on $T^*M \setminus 0$, we define $\Phi(x,\xi) : T^*M \setminus 0 \to (0,\infty)$ by

$$\varphi(x,\xi) = \frac{1}{2}\Phi(x,\xi)^2. \tag{2.9}$$

Clearly, $\Phi$ is $C^\infty$ on $T^*M \setminus 0$, and, parallel to (1.2),

$$\Phi(x,\lambda\xi) = \lambda\Phi(x,\xi) \quad \forall\, \xi \in T^*M \setminus 0,\ \lambda \in (0,\infty). \tag{2.10}$$

We also claim the following analogue of (1.4) holds:

$$D_\xi^2 \varphi(x,\xi) \;\text{ is positive definite,} \tag{2.11}$$

as a symmetric bilinear form on $T^*M$, for each $x \in M$ and each nonzero $\xi \in T_x^*M$. In other words, $\Phi(x,\xi)$ produces a Minkowski norm on each space $T_x^*M$. To see this, write

$$\begin{aligned} D_\xi^2 \varphi(x,\xi) &= D_\xi v(x,\xi) \\ &= \big(D_v \xi(x,v)\big)^{-1} \\ &= \big(D_v^2 f(x,v)\big)^{-1}, \end{aligned} \tag{2.12}$$

where the first identity follows by differentiating (2.5), the second by the chain rule, and the third by (2.4).

## 3 Finsler Symbols, Pseudodifferential Operators, and Hyperbolic PDEs

A smooth function $p(x,\xi)$ on $T^*M \setminus 0$, homogeneous of degree $m$ in $\xi$, is the principal symbol of a pseudodifferential operator $p(x,D)$, uniquely determined up to an operator of degree $m-1$, given in local coordinates by a Fourier integral representation:

$$p(x,D)u = (2\pi)^{-n} \iint p(x,\xi)u(y)e^{i(x-y)\cdot\xi}\,dy\,d\xi. \tag{3.1}$$

Here, $p(x,\xi)$ is possibly regularized for small $|\xi|$. One says $p(x,D)$ is elliptic if $|p(x,\xi)| \geqslant C|\xi|^m$ for large $|\xi|$, and strongly elliptic if $\operatorname{Re} p(x,\xi) \geqslant C|\xi|^m$ for large $|\xi|$. In particular, a Finsler symbol $\Phi(x,\xi)$ is the principal symbol of a special sort of strongly elliptic, first order, pseudodifferential operator, which can furthermore be taken to be self-adjoint.

Real-valued, first order symbols arise in the analysis of strictly hyperbolic PDEs, as we now briefly describe. Let

$$P(x,D_t,D_x) = D_t^m + \sum_{k=0}^{m-1} A_k(x,D_x)D_t^k \tag{3.2}$$

be a differential operator of order $m$, in this case a positive integer. (One could have $t$-dependent coefficients, but for simplicity we take coefficients independent of $t$.) Here, $D_t = (1/i)\partial_t$, $D_x = (1/i)\partial_x$, and $A_k(x,D_x)$ is a differential operator of order $m-k$. The principal symbol of $P(x,D_t,D_x)$ is a polynomial in $(\tau,\xi)$, homogeneous of order $m$:

$$P_m(x,\tau,\xi) = \tau^m + \sum_{k=0}^{m-1} A_k^b(x,\xi)\tau^k. \tag{3.3}$$

The operator (3.2) is said to be *strictly hyperbolic* if for each $\xi \neq 0$ the symbol $P_m$ has $m$ roots $\tau = \lambda_k(x,\xi)$, all real and distinct, so

$$\begin{gathered} P_m(x,\tau,\xi) = (\tau - \lambda_1(x,\xi)) \cdots (\tau - \lambda_m(x,\xi)), \\ \lambda_1(x,\xi) < \cdots < \lambda_m(x,\xi). \end{gathered} \tag{3.4}$$

The functions $\lambda_k(x,\xi)$ are smooth on $T^*M \backslash 0$ and homogeneous of degree 1 in $\xi$, hence are the principal symbols of first order pseudodifferential operators. The analysis of solutions to the partial differential equations $P(x, D_t, D_x)u = 0$ is essentially equivalent to the analysis of the pseudodifferential evolution equations

$$\frac{\partial u}{\partial t} = i\lambda_k(x,D)u. \tag{3.5}$$

In either case, short-time approximate solution operators (known as parametrices) can be constructed in the form

$$S_k(t)f = (2\pi)^{-n} \iint f(y) a_k(t,x,y,\xi) e^{i\theta_k(t,x,y,\xi)}\, dy\, d\xi. \tag{3.6}$$

The phase functions $\theta_k$ solve eikonal equations:

$$\frac{\partial \theta_k}{\partial t} = \lambda_k(x, d_x\theta_k), \quad \theta_k(0,x,y,\xi) = (x-y)\cdot\xi, \tag{3.7}$$

and the amplitudes $a_k(t,x,y,\xi)$ are symbols of order zero, with asymptotic expansions whose terms solve a succession of transport equations. For details one can see [9, Chapt. 8].

The standard Hamilton–Jacobi approach to solving (3.7) brings in the flow generated by the Hamiltonian vector field $H_{\lambda_k}$. In particular, when the symbol $\lambda_k(x,\xi)$ (or its negative) is elliptic and $\lambda_k(x,\xi)^2$ is strongly convex, the integral curves of this flow correspond to geodesics of the Finsler metric associated to $\lambda_k(x,\xi)$, by the correspondence defined in Sect. 2.

The flow generated by $H_{\lambda_k}$ defines the way the solution operator $S_k(t)$ in (3.6) moves singularities of $f$. More precisely, $f$ has a "wave front set," $\mathrm{WF}(f)$, a closed, conic subset of $T^*M \setminus 0$, whose projection onto $M$ is the singular support of $f$, and the wave front set of $S_k(t)f$ is obtained from $\mathrm{WF}(f)$ by applying the flow generated by $H_{\lambda_k}$. (Details can be found in [9, Chapt. 8], [10, Chapt. 7], or [4, Vol. 3].) In particular, if $\lambda_k(x,\xi)$ is a Finsler symbol, and if $f = \delta_p$, the point mass concentrated at $p \in M$, then the singular support of $e^{it\lambda_k(x,D)}\delta_p$ is equal to the "sphere" $\Sigma_{|t|}(p)$, of points of Finsler distance $|t|$ from $p$, for small $|t|$ (with modifications once caustics form).

The symbols $\lambda(x,\xi)$ that can arise in (3.4) from hyperbolic PDEs need not be elliptic, and even if such $\lambda(x,\xi)$ is elliptic, $\lambda(x,\xi)^2$ need not be strongly convex. However, when these properties hold, the wave propagation has particularly nice properties. One indication of this, the tame behavior of $e^{it\lambda(x,D)}\delta_p$, for small $|t|$, was described above. The operator $\lambda(x,D)$ (typically taken to be self-adjoint) also has nicer spectral properties when $\lambda(x,\xi)$ is a Finsler symbol than one finds in more general cases. We mention one example here. The following result is Theorem 5.2.1 in [8], though the term "Finsler symbol" was not used there.

**Proposition 3.1.** *Let $A$ be a first order, self-adjoint pseudodifferential operator on a compact manifold $M$, whose principal symbol is a Finsler symbol. Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be an orthonormal basis of eigenfunctions of $A$, $A\varphi_k = \lambda_k\varphi_k$, and consider the Riesz means of order $\delta$:*

$$\Sigma_R^\delta f(x) = \sum_{\lambda_k \leqslant R} \Big(1 - \frac{\lambda_k}{R}\Big)^\delta \widehat{f}(k)\varphi_k, \tag{3.8}$$

*where $\widehat{f}(k) = (f,\varphi_k)_{L^2}$. Then, given*

$$p \in \Big[1, 2\frac{n+1}{n+3}\Big] \bigcup \Big[2\frac{n+1}{n-1}, \infty\Big), \tag{3.9}$$

*and*

$$\delta > \delta(p) = \max\Big\{n\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{2}, 0\Big\}, \tag{3.10}$$

*one has*

$$\Sigma_R^\delta f \longrightarrow f \quad \text{in } L^p\text{-norm as} \quad R \to \infty \quad \forall\, f \in L^p(M), \tag{3.11}$$

*and more generally $\Sigma_R^\delta f \to f$ in $L^q$-norm for all $f \in L^q(M)$, as long as $q \in [p,p']$ (and $q \neq \infty$).*

*Remark 3.1.* Since $A$ in Proposition 3.1 is elliptic, if $p \in (1,\infty)$,

$$(A^2+1)^{-s/2} : L^p(M) \to H^{s,p}(M), \ \text{isomorphically}, \tag{3.12}$$

where $H^{s,p}(M)$ is the $L^p$-Sobolev space. Since these operators commute with $\Sigma_R^\delta$, we have the following corollary. In the setting of Proposition 3.1, if $p$ and $\delta$ satisfy (3.9)–(3.10) and $p > 1$, then

$$\Sigma_R^\delta f \longrightarrow f \quad \text{in} \quad H^{s,p}\text{-norm as} \quad R \to \infty \quad \forall\, f \in H^{s,p}(M), \tag{3.13}$$

and more generally such convergence holds in $H^{s,q}$-norm, for $f \in H^{s,q}(M)$ provided that $q \in [p,p']$.

We next present a simple class of examples of fourth order, strictly hyperbolic PDEs that give rise to non-Riemannian Finsler symbols. Namely, we consider operators whose symbols have the form

$$P_a(\tau,\xi) = \big(\tau^2 - g(\xi)\big)\big(\tau^2 - h(\xi)\big) + aq(\xi). \tag{3.14}$$

Here, $g(\xi)$ and $h(\xi)$ are positive definite quadratic forms:

$$g(\xi) = g^{jk}\xi_j\xi_k, \quad h(\xi) = h^{jk}\xi_j\xi_k, \tag{3.15}$$

$q(\xi)$ is a real-valued polynomial in $\xi$, homogeneous of degree 4, and $a \in \mathbb{R}$. We assume that

$$g(\xi) - h(\xi) \geqslant C|\xi|^2 \tag{3.16}$$

for some $C > 0$. Then the roots $\tau = \lambda(\xi)$ of $P_a$ satisfy

$$\tau^2 = \lambda(\xi)^2 = \frac{g(\xi)+h(\xi)}{2} \pm \frac{1}{2}\sqrt{\big(g(\xi)-h(\xi)\big)^2 - 4aq(\xi)}. \tag{3.17}$$

It is clear that as long as $a \in \mathbb{R}$ is sufficiently small one has four distinct, real roots, which are small perturbations of the 4 roots at $a = 0$:

$$-\sqrt{g(\xi)} < -\sqrt{h(\xi)} < \sqrt{h(\xi)} < \sqrt{g(\xi)}. \tag{3.18}$$

It is also clear that as long as $a \in \mathbb{R}$ is small enough, then each $\lambda(\xi)^2$ is strongly convex.

By contrast, here is a construction of some symbols of third order, strictly hyperbolic PDE, at least one of whose roots is a non-Finsler symbol. Namely, we consider symbols of the form

$$Q_a(\tau,\xi) = \tau\big(\tau^2 - g(\xi)\big) + aq(\xi). \tag{3.19}$$

Here, $g(\xi)$ is a positive definite quadratic form, as in (3.15), and $q(\xi)$ is a real-valued polynomial in $\xi$, homogeneous of degree 3. At $a = 0$, the three roots are

$$-\sqrt{g(\xi)} < 0 < \sqrt{g(\xi)}, \tag{3.20}$$

and for small real $a$ we get real roots that are small perturbations of these. A calculation shows that $\lambda_2(\xi) = 0$ perturbs to

$$\lambda_2(\xi) = a\frac{q(\xi)}{g(\xi)} + O(a^2|\xi|). \tag{3.21}$$

Thus, for small $a$, $\lambda_2(\xi)$ is a small perturbation of a constant multiple of $q(\xi)/g(\xi)$. For example, we can pick $q(\xi) = \xi_2^3$ and $g(\xi) = |\xi|^2$, and get small perturbations of

$$\lambda(\xi) = \frac{\xi_2^3}{|\xi|^2} = \frac{\xi_2^3}{\xi_1^2+\xi_2^2}, \quad \text{if } n = 2. \tag{3.22}$$

Figure 1 represents a graph of the singular support of $e^{it\lambda(D)}\delta$, for some fixed $t > 0$, when $\lambda(\xi)$ is given by (3.22).

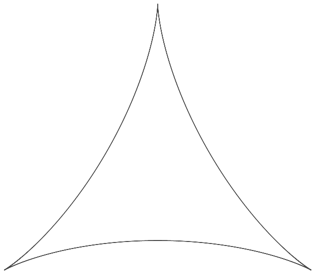

**Fig. 1** Singular support of $e^{it\lambda(D)}\delta$, $\lambda(\xi) = |\xi|^{-2}\xi_2^3$.

The singular supports for different values of $t$ are dilates of each other. The three cusps form immediately, in contrast to the behavior of $e^{itA}\delta_p$ when $A$ has a Finsler principal symbol.

The symbol $\lambda(\xi)$ in (3.22) is also nonelliptic. The following is an elliptic, non-Finsler symbol:

$$\lambda(\xi) = \frac{\xi_2^3}{|\xi|^2} + \frac{11}{10}|\xi|. \tag{3.23}$$

Figure 2 shows the singular support of $e^{it\lambda(D)}\delta$ for such $\lambda(\xi)$, at a fixed $t > 0$.

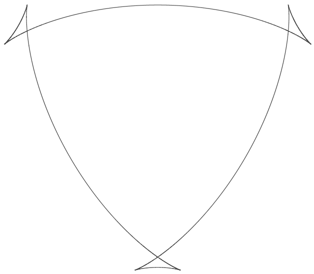

**Fig. 2** Singular support of $e^{it\lambda(D)}\delta$, $\lambda(\xi) = |\xi|^{-2}\xi_2^3 + (11/10)|\xi|$.

# 4 Katok's Construction and Its Harmonic Analysis Counterpart

Katok [5] constructed Finsler metrics on the sphere $S^n$ whose geodesic flows have remarkable properties, further explored in [13]. Here, we recall these metrics, bring in the associated pseudodifferential operators, and discuss a problem in harmonic analysis that arises in investigating these operators.

The Finsler metrics that arise here belong to a special class known as "Randers metrics." Generally, a Randers metric on a manifold $M$ has the form

$$F(x,v) = g_x(v,v)^{1/2} + g_x(v,X), \tag{4.1}$$

where $g_x$ is a Riemannian metric tensor on $M$ and $X$ a real vector field on $M$, satisfying $g_x(X,X) < 1$. (This condition on $X$ is necessary and sufficient for strong convexity (1.4).) In such a case, the correspondence (1.9) yields a "Randers symbol" on $T^*M$, having the form

$$\Phi(x,\xi) = h_x(\xi,\xi)^{1/2} + \langle Y,\xi\rangle = \lambda(x,\xi) + \eta(x,\xi), \tag{4.2}$$

where $h_x$ is a positive definite quadratic form on $T_x^*M$ and $Y$ a real vector field on $M$, satisfying $\langle Y,\xi\rangle = h_x(y,\xi),\ h_x(y,y) < 1$. For the reader convenience we sketch the calculation establishing this fact in Appendix (Sect. 5). If $X \neq 0$, then $h_x$ is not the form on $T_x^*M$ dual to $g_x$.

The symbol (4.2) is the symbol of a pseudodifferential operator of the form

$$A = \sqrt{-\Delta + c} - iY = \Lambda - iY, \tag{4.3}$$

where $\Delta$ is the Laplace–Beltrami operator on $M$ for the metric tensor dual to $h$, and $c$ is a nonnegative constant. One can take $c = 0$, but sometimes another choice of $c$ will be more interesting.

The Katok examples are most naturally described by directly specifying $\Phi(x,\xi)$. Let $h_x$ be the quadratic form on $T^*S^n$ dual to the standard metric tensor (which we denote $g$) on the unit sphere $S^n$, and let $Y$ generate a rotation. We require $g_x(Y,Y) < 1$ on $S^n$. We assume furthermore that

$$Y = \alpha Y_0, \quad -1 < \alpha < 1, \tag{4.4}$$

where $Y_0$ generates a periodic group of rotations $R_0(t)$, of minimum period $2\pi$. (If $n = 2,\ Y$ necessarily has this form.)

The analysis of the geodesic flow of such a Finsler metric is equivalent to the analysis of the flow generated by $H_\Phi$, with $\Phi$ given by (4.2). In the present case, this is simplified by the fact that $H_\Phi = H_\lambda + H_\eta$, and the vector fields $H_\lambda$ and $H_\eta$ commute; hence their flows commute. In other words, if $\mathcal{F}_\Phi^t$ is the flow generated by $H_\Phi$, then, with obvious notation,

$$\mathcal{F}_\Phi^t = \mathcal{F}_\eta^t \circ \mathcal{F}_\lambda^t. \tag{4.5}$$

Now, $\mathcal{F}_\lambda^t$ corresponds to the geodesic flow on the standard sphere $S^n$, which is very well understood. Meanwhile $\mathcal{F}_\eta^t$ is simply the flow on $T^*S^n$ induced by the action of $R_0(\alpha t)$ on $S^n$, and that is also a straightforward object. Putting these together, we certainly have some good hold of the flow generated by $H_\Phi$.

Here is one notable phenomenon. The flow generated by $H_\lambda$ is "perfectly focusing" at times $t = k\pi,\ k \in \mathbb{Z}$. For $k$ odd one has geodesics from $p$ focusing at the antipodal point $-p \in S^n$, and for $k$ even the geodesics focus back at

$p$. Now, for the flow generated by $H_\Phi$, we see that orbits arising over $p \in S^n$ focus at time $t = \pi$ at $R_0(\pi\alpha)(-p)$. If $R_0(t)$ leaves $p$ and $-p$ invariant, this point is $-p$. Otherwise, it is some other point, as long as $\alpha \in (-1, 1)$, but $\alpha \neq 0$. One gets a second focusing, at time $2\pi$ and location $R_0(2\pi\alpha)p$, and generally a $k$th focusing, at time $k\pi$ and location $R_0(k\pi\alpha)((-1)^k p)$.

An obvious dichotomy arises according to whether $\alpha \in (-1, 1)$ is rational or irrational. In the former case, the flow $\mathcal{F}_\Phi^t$ is periodic, and in the latter case it is not. Part of the thrust of [5] and [13] lay in showing that, in the latter case, $\mathcal{F}_\Phi^t$ has remarkably few closed orbits. We refer to these papers for more on the dynamical properties of $\mathcal{F}_\Phi^t$, and turn our attention to the spectral properties of the operator $A$, given by (4.3).

In the case we are considering, with $\Delta$ the Laplace–Beltrami operator on the standard sphere $S^n$, it is natural to take $c$ in (4.3) to be

$$c = \Big(\frac{n-1}{2}\Big)^2. \tag{4.6}$$

This produces the following pleasant result (see. for example, [10, Chapt. 8, Sect. 4]):

$$\text{Spec } \Lambda = \Big\{\frac{n-1}{2} + k : k = 0, 1, 2, \dots\Big\}. \tag{4.7}$$

Also, under the hypothesis (4.4) on $Y$, we have

$$\text{Spec}\,(-iY) = \alpha\mathbb{Z} \tag{4.8}$$

since Spec $iY_0 = \mathbb{Z}$.

Furthermore, the operators $\Lambda = \sqrt{-\Delta + c}$ and $iY$ commute. This fact carries more information than the fact that the Hamiltonian vector fields $H_\lambda$ and $H_\eta$ commute; the latter result is equivalent to the statement that the commutator $[\Lambda, iY]$ is a pseudodifferential operator of order zero. Since $\Lambda$ and $iY$ commute, $iY$ preserves each eigenspace of $\Lambda$, and we have

$$\text{Spec } A \subset \text{Spec } \Lambda + \text{Spec}\,(-iY) = \Big(\frac{n-1}{2} + \mathbb{Z}^+\Big) + \alpha\mathbb{Z}. \tag{4.9}$$

Also we have

$$e^{itA} = e^{it\Lambda}e^{tY}. \tag{4.10}$$

The dichotomy of $\alpha$ being rational or irrational is seen here to be manifested on an operator level. The rational case yields periodicity:

$$\alpha = \frac{\mu}{\nu} \in \mathbb{Q} \Longrightarrow \text{Spec } 2\nu A \subset \mathbb{Z} \Longrightarrow e^{i(t+4\pi\nu)A} = e^{itA}, \tag{4.11}$$

while the irrational case does not.

In the remainder of this section, we discuss how the dynamical and operator theoretical results described above bear on the question of convergence of the eigenfunction expansion of a function $f$ on $S^n$, in terms of the eigenfunctions of the operator $A$ given by (4.3), with $Y$ given by (4.4). As we will see, separate analyses are required according to whether $\alpha$ is rational or irrational.

The object of our study is

$$S_R f(x) = \sum_{\lambda_j \leqslant R} (f, \varphi_j)\, \varphi_j(x), \tag{4.12}$$

where $\{\varphi_j : j \geqslant 0\}$ is an orthonormal basis of $L^2(S^n)$ consisting of eigenfunctions of $A$, $A\varphi_j = \lambda_j \varphi_j$. We can write (4.12) as

$$S_R f = \chi_R(A) f, \tag{4.13}$$

where $\chi_R(\lambda) = 1$ for $|\lambda| \leqslant R,\ 0$ for $|\lambda| > R$. (It is sometimes convenient to set $\chi_R(\lambda) = 1/2$ for $|\lambda| = 1$, and to adjust (4.12) accordingly.)

A broad class of functions $\psi(A)$ of the operator $A$ can be analyzed as

$$\psi(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\psi}(t) e^{itA}\, dt, \tag{4.14}$$

where

$$\widehat{\psi}(t) = \int_{-\infty}^{\infty} \psi(\lambda) e^{-i\lambda t}\, d\lambda. \tag{4.15}$$

In particular, (4.13) can be rewritten as

$$S_R f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\, e^{itA} f(x)\, dt. \tag{4.16}$$

Applications of such a formula to the pointwise behavior of $S_R f(x)$ as $R \to \infty$ have been given in [7, 1, 2, 11, 12], amongst other places. The emphasis has been on $A = \sqrt{-\Delta + c}$ in these papers, but many of the results carry over in a straightforward manner to $A$ of the form (4.3).

A good understanding of the behavior of $u(t,x) = e^{itA} f(x)$, which satisfies an evolution equation of the form (3.5), leads to results on the behavior of (4.16) as $R \to \infty$. This works neatly when $u(t,x)$ is a wave on $\mathbb{R} \times M$, where $M$ is noncompact and the waves scatter off to infinity. When $M$ is compact, as it is here ($M = S^n$) the fact that the range of integration in (4.16) is $t \in (-\infty, \infty)$ requires further work. The case $A = \Lambda$ was handled in [7, Sect. 6], by the following device. When $A = \Lambda$, the spectrum, given by (4.7), consists of integers (if $n$ is odd) or half-integers (if $n$ is even), so we have periodicity:

$$e^{i(t+2\pi\nu)A} = e^{itA}, \tag{4.17}$$

with $\nu = 1$ for odd $n$ and $\nu = 2$ for even $n$. In such a case, one can replace (4.14) by

$$\psi(A) = \frac{1}{2\pi\nu} \int_{\mathbb{R}/(2\pi\nu\mathbb{Z})} \widehat{\psi}(t) e^{itA}\, dt, \tag{4.18}$$

where now

$$\widehat{\psi}(t) = \sum_{k=-\infty}^{\infty} \psi\Big(\frac{k}{\nu}\Big) e^{-ikt/\nu}. \tag{4.19}$$

In particular, with $\psi(k/\nu) = \chi_R(k/\nu)$, we have

$$\widehat{\psi}(t) = \sum_{k=-R\nu}^{R\nu} e^{-ikt/\nu} = \frac{\sin\big(R + \frac{1}{2\nu}\big)t}{\sin \frac{t}{2\nu}}, \tag{4.20}$$

at least as long as $R\nu$ is an integer. Thus, we can replace (4.16) by

$$S_R f(x) = \frac{1}{2\pi\nu} \int_{\mathbb{R}/(2\pi\nu\mathbb{Z})} \frac{\sin\big(R + \frac{1}{2\nu}\big)t}{\sin \frac{t}{2\nu}}\, e^{itA} f(x)\, dt. \tag{4.21}$$

Now, we are integrating over $t$ on a circle, which is compact, so local analysis of waves suffices to treat the behavior of (4.21) as $R \to \infty$ (with $R\nu$ integral).

Formula (4.21) works more generally, as long as $A$ has the form (4.3)–(4.4) with $\alpha$ rational, as a consequence of (4.11) (we might have to double $\nu$). Thus, the analysis in [7, Sect. 6] applies with little change to this more general situation.

To illustrate the results, let us take $n = 3$, pick $p \in S^3$, $a \in (0, \pi)$, and let

$$\begin{aligned} f(x) = 1, &\quad \operatorname{dist}(x,p) < a, \\ \tfrac{1}{2}, &\quad \operatorname{dist}(x,p) = a, \\ 0, &\quad \operatorname{dist}(x,p) > a. \end{aligned} \tag{4.22}$$

In the case $\alpha = 0$, analyzed in [7, Sect. 6], one has pointwise convergence $S_R f(x) \to f(x)$ for all $x \in S^3$, with two exceptions, namely $x = p$ and $x = -p$. At these two points, $S_R f(p)$ and $S_R f(-p)$ exhibit an oscillatory divergence. This type of behavior, discovered first for Euclidean space Fourier inversion when $f$ is the characteristic function of a ball in $\mathbb{R}^3$ by Pinsky, is called the *Pinsky phenomenon*. The analysis of this Pinsky phenomenon as carried out in [7] involves the analysis of the focusing of the wave $u(t,x) = e^{itA} f(x)$. An analysis of the behavior, valid uniformly in a neighborhood of such a focus, is contained in results in [12, Sect. 8]. One also has a Gibbs phenomenon, on a neighborhood of the sphere $\{x : \operatorname{dist}(x,p) = a\}$, analyzed (in a more general context) in [7, Sect. 11].

In the more general situation where $A$ is given by (4.3)–(4.4) with $\alpha$ rational, the same results hold, except that now the Pinsky phenomenon is manifested at all the points $R_0(k\pi\alpha)((-1)^k p)$ of focusing for the wave $u(t,x) = e^{itA}f(x)$. For $\alpha$ rational this is a finite set of points in $S^3$. There are analogous results for other functions $f$, and also results in other dimensions, as one can see by adapting results in [7] and [11, 12].

If $\alpha$ is irrational, the periodicity (4.17) fails. Such a situation also holds for the analogue of $S_R f = \chi_R(A)f$ when $A = \sqrt{-\Delta + c}$ on a typical compact Riemannian manifold. These cases are harder to analyze when the compactification trick is not available, but some successful techniques have been developed. An approach initiated in [1] and further developed in [2] and [11, 12], amongst other places, involves breaking $S_R f$ into two pieces:

$$S_R f(x) = S_R^\beta f(x) + T_R^\beta f(x), \tag{4.23}$$

where we pick $\beta \in C_0^\infty(\mathbb{R})$ with $\beta(t) = 0$ for $|t| \leqslant a,\ 0$ for $|t| \geqslant a+1$ (given some $a > 0$) and define

$$S_R^\beta f(x) = \frac{1}{\pi} \int \frac{\sin Rt}{t}\, \beta(t)\, e^{itA} f(x)\, dt. \tag{4.24}$$

The following result is Proposition 1 of [11]. In that paper, we took $A = \sqrt{-\Delta}$, but the analysis is the same for any positive, self-adjoint elliptic pseudodifferential operator of order 1, on a compact manifold $M$, of dimension $n$.

**Proposition 4.1.** *Fix $x \in M$ and assume that, for $R > 1,\ \varepsilon \in (0,1]$,*

$$\sum_{R \leqslant \lambda_j \leqslant R+\varepsilon} |\varphi_j(x)|^2 \leqslant \delta(\varepsilon) R^{n-1} + \gamma(\varepsilon, R) R^{n-1}, \tag{4.25}$$

*with*

$$\lim_{R\to\infty} \gamma(\varepsilon, R) = 0 \quad \forall\ \varepsilon > 0; \quad \lim_{\varepsilon\to 0} \delta(\varepsilon) = 0. \tag{4.26}$$

*Assume that $f \in \Pi^{-(n-3)/2}(M)$, i.e.,*

$$\sum_{R \leqslant \lambda_j \leqslant R+1} |(f, \varphi_j)|^2 \leqslant C R^{-(n-1)}. \tag{4.27}$$

*Furthermore, assume that there exists $T_0 \in (0, \infty)$ such that $u(t,x) = e^{itA} f(x)$ satisfies*

$$u(\cdot, x) \in L^1_{\text{loc}}(\mathbb{R} \setminus [-T_0, T_0]). \tag{4.28}$$

*Then for each $\beta \in C_0^\infty(\mathbb{R})$ such that $\beta(t) = 1$ for $|t| \leqslant T_0 + 1$, we have*

$$\lim_{R\to\infty} \left| S_R f(x) - S_R^\beta f(x) \right| = 0. \tag{4.29}$$

There are three conditions to verify, and we claim they can be verified when $M = S^3,\ A$ is given by (4.3)–(4.4), and $f$ is given by (4.22) (with some exceptions, when $R_0(t)$ is given by (4.38), as we will discuss below). First we address the issue of when (4.25) holds. The following result, inspired by [3], is Proposition 2 of [11].

**Proposition 4.2.** *Fix $x \in M$. Assume that the Hamilton flow-out (via $H_\Phi$, where $\Phi$ is the principal symbol of $A$) of $T^*_x M \setminus 0$ lies over $x$ for only a discrete set of times $t$. Assume that, except for $t = 0$, any caustic of this flow-out lying over $x$ has order $< (n-1)/2$. Then we have*

$$\sum_{R \leqslant \lambda_j \leqslant R+\varepsilon} |\varphi_j(x)|^2 \leqslant C\varepsilon R^{n-1} + \gamma(\varepsilon, R) R^{n-1}, \tag{4.30}$$

*for $\varepsilon \in (0,1]$, $R \in (1,\infty)$, with*

$$\lim_{R\to\infty} \gamma(\varepsilon, R) = 0 \quad \forall\ \varepsilon \in (0,1]. \tag{4.31}$$

Let us return to the setting $M = S^3$, and let $A$ be given by (4.3)–(4.4). Let us suppose that $R_0(t)$ acts on $S^3 \subset \mathbb{R}^4$ as

$$R_0(t) = \begin{pmatrix} \rho_0(t) & \\ & \rho_0(t) \end{pmatrix}, \quad \rho_0(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{4.32}$$

In such a case, the hypotheses of Proposition 4.2 hold for each $x \in S^3$, so (4.25)–(4.26) hold. As for (4.27), it is a general fact that if $M$ is a compact Riemannian manifold and $\Omega \subset M$ is a smoothly bounded set, then

$$\chi_\Omega \in \Pi^0(M), \quad \text{i.e.,} \quad \sum_{R\leqslant \lambda_j \leqslant R+1} |(\chi_\Omega, \varphi_j)|^2 \leqslant C R^{-2}. \tag{4.33}$$

This is a special case of results proved in [2] (see also [11, Sect. 4]). This yields (4.27), when $n = 3$.

As for (4.28), when $f$ is given by (4.22), then for $x_k = R_0(k\pi\alpha)((-1)^k p)$ one sees that $u(\cdot, x_k)$ will not be $L^1$ in an interval about $t_k^\pm = \pm k\pi$, due to focusing, but $u(\cdot, x_k) \in L^1_{\text{loc}}(\mathbb{R} \setminus \{-k\pi, k\pi\})$ (if $\alpha$ is irrational), since there will not be any other focusing at $x_k$. (If $\alpha$ is rational, there will be other focusing at $x_k$, infinitely many times, but the previous analysis has taken care of the behavior of $S_R f(x)$ in that case.) If $x \notin \{x_k : k \in \mathbb{Z}\}$, then $u(\cdot, x) \in L^1_{\text{loc}}(\mathbb{R})$. Consequently, (4.28) holds for all $x \in S^3$ when $f$ is given by (4.22), as long as $\alpha$ is irrational. Thus, the conclusion (4.29) holds, and the pointwise behavior of $S_R f(x)$ as $R \to \infty$ is controlled by the behavior of $S_R^\beta f(x)$, with

$$\beta \in C_0^\infty(\mathbb{R}), \quad \beta(t) = 1 \ \text{ for } \ |t| \leqslant T_0(x) + 1, \tag{4.34}$$

and with $T_0(x)$ as described above.

As mentioned, the methods of [7] apply to $S_R^\beta f(x)$. The behavior is as follows. Let

$$\mathcal{O}(p) = \{R_0(k\pi\alpha)((-1)^k p) : k \in \mathbb{Z}\}. \tag{4.35}$$

For $\alpha$ irrational and $R_0(t)$ given by (4.32), this set is dense in the circle

$$\mathcal{C}(p) = \{R_0(t)p : t \in \mathbb{R}\}. \tag{4.36}$$

We have pointwise convergence

$$S_R f(x) \to f(x) \quad \forall\, x \notin \mathcal{O}(p). \tag{4.37}$$

On the other hand, if $x = x_k \in \mathcal{O}(p)$ and $\beta$ satisfies (4.34), then the Pinsky phenomenon is manifested for $S_R^\beta f(x)$. Since (4.29) holds, this implies that $S_R f(x_k)$ does not converge to $f(x_k)$, but has an oscillatory divergence.

Results are similar, up to a point, when instead of (4.32) one has

$$R_0(t) = \begin{pmatrix} \rho_0(t) & \\ & I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.38}$$

In this case, the points in

$$\mathcal{F} = \{(0,0,a,b)^t \in \mathbb{R}^4 : a^2 + b^2 = 1\} \tag{4.39}$$

are fixed points for the action of $R_0$. The compactification trick still works if $\alpha \in \mathbb{Q}$, so from here on we concentrate on the case $\alpha \notin \mathbb{Q}$. In this case, the hypotheses of Proposition 4.2 hold for $x \in S^3$ if and only if $x \notin \mathcal{F}$. Hence (4.25) holds for $x \in S^3 \setminus \mathcal{F}$, but we do not have this result for $x \in \mathcal{F}$. To be sure, (4.27) still holds, if $f$ is given by (4.22). In such a case, (4.28) holds unless

$$p \in \mathcal{F} \ \text{ and } \ x \in \mathcal{O}(p), \tag{4.40}$$

where $\mathcal{O}(p)$ is given by (4.35), but we do not have (4.28) if (4.40) holds. (Note that $p \in \mathcal{F} \Rightarrow \mathcal{O}(p) = \{\pm p\}$.) Hence, in this situation, we conclude that

$$x \in S^3 \setminus \mathcal{F}, \ \ x \notin \mathcal{O}(p) \Longrightarrow S_R f(x) \to f(x), \tag{4.41}$$

as $R \to \infty$, when $f$ is given by (4.41). Furthermore, the Pinsky phenomenon is manifested if $x \in S^3 \setminus \mathcal{F}$ and $x \in \mathcal{O}(p)$. However, if $x \in \mathcal{F}$, then Proposition 4.1 is not applicable, and at this point the behavior of $S_R f(x)$ as $R \to \infty$ is not known (at least, not to this author).

*Remark 4.1.* When $A$ is given by (4.3)–(4.4) and $\Delta$ is the Laplace–Beltrami operator on $S^n$, then the eigenfunction expansions (4.12) are expansions in spherical harmonics, whatever the value of $\alpha$. However, the order in which these spherical harmonics enter the expansion is strongly affected by the choice of $\alpha$. One might compare this situation with phenomena discussed in [12, Sect. 10].

*Remark 4.2.* As noted in [13], the Katok construction can be extended, replacing $S^n$ by any compact, rank-one symmetric space, such as a complex or quaternionic projective space. It turns out that spectral behavior similar to (4.7) also holds for these other cases, and our analysis of $S_R f(x)$ can be carried out in this greater generality. See [7, Sect. 7] for the case $\alpha = 0$.

## 5 Appendix. Randers–Randers Duality

Here, we show that if $\Phi(x, \xi)$ is a Finsler symbol of Randers type, then the associated Finsler metric $F(x, v)$ is also of Randers type (and conversely). This recovers part of Theorem 7.4.3 in [6].

There is no need to record $x$ dependence, and we can consider without loss of generality $\varphi(\xi) = \Phi(\xi)^2/2$, of the form

$$\varphi(\xi) = \frac{1}{2}\big(|\xi| + b \cdot \xi\big)^2 = \frac{1}{2}|\xi|^2 + \frac{1}{2}(b \cdot \xi)^2 + |\xi|(b \cdot \xi), \tag{5.1}$$

with $|\xi|^2 = \xi \cdot \xi,\ |b| < 1$. A calculation gives

$$v = D_\xi \varphi(\xi) = \big(|\xi| + b \cdot \xi\big)\Big(b + \frac{\xi}{|\xi|}\Big) = \sqrt{2\varphi(\xi)}\Big(b + \frac{\xi}{|\xi|}\Big). \tag{5.2}$$

Our task is to write $f(v) = \varphi(\xi)$ explicitly as a function of $v$. To begin, we have from (5.2) that

$$b + \frac{\xi}{|\xi|} = \frac{v}{\sqrt{2f(v)}}. \tag{5.3}$$

Subtracting $b$ gives a vector of length one, hence the identity

$$1 = \Big(\frac{v}{\sqrt{2f(v)}} - b\Big) \cdot \Big(\frac{v}{\sqrt{2f(v)}} - b\Big) = \frac{|v|^2}{2f(v)} - 2\frac{b \cdot v}{\sqrt{2f(v)}} + |b|^2. \tag{5.4}$$

This gives for $F(v) = \sqrt{2f(v)}$ the quadratic equation

$$(1 - |b|^2)F^2 + 2(b \cdot v)F - |v|^2 = 0, \tag{5.5}$$

and hence

$$F(v) = -\frac{b \cdot v}{1 - |b|^2} + \frac{1}{1 - |b|^2}\sqrt{(b \cdot v)^2 + (1 - |b|^2)|v|^2}. \tag{5.6}$$

In other words,

$$F(v) = \sqrt{v \cdot Qv} + c \cdot v, \tag{5.7}$$

where

$$c = -\frac{b}{1 - |b|^2} \tag{5.8}$$

and

$$v \cdot Qv = \frac{(1 - |b|^2)|v|^2 + (b \cdot v)^2}{(1 - |b|^2)^2}. \tag{5.9}$$

The strong convexity of $f(v)$ is already known, so we see that $F(v)$ is of Randers type.

**Acknowledgment.** Thanks to Robert Bryant for a stimulating Colloquium, introducing Finsler geometry, and in particular for exposition of the Katok examples.

The author was supported in part by NSF (grant no. DMS-0456861).

## References

1. Brandolini, L., Colzani, L.: Localization and convergence of eigenfunction expansions. J. Fourier Anal. Appl. **5**, 431-447 (1999)
2. Brandolini, L., Colzani, L.: Decay of Fourier transforms and summability of eigenfunction expansions. Ann. Scuola Norm. Sup. Pisa **29**, 611-638 (2000)
3. Duistermaat, J.J., Guillemin, V.: The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. **29**, 39-79 (1975)
4. Hörmander, L.: The Analysis of Linear Partial Differential Operators. Vols. 3-4. Springer-Verlag, New York (1985)
5. Katok, A.B.: Ergodic perturbations of degenerate integrable Hamiltonian systems (in Russian). Izv. Akad. Nauk SSSR, Ser. Mat. **37**, 539-576 (1973); English transl.: Math. USSR Izv. **7**, 535-571 (1974)
6. Miron, R., Hrimiuc, D., Shimada, H., Sabau, S.: The Geometry of Hamilton and Lagrange Spaces. Kluwer, Boston (2001)
7. Pinsky, M., Taylor, M.: Pointwise Fourier inversion: a wave equation approach. J. Fourier Anal. Appl. **3**, 647-703 (1997)
8. Sogge, C.: Fourier Integrals in Classical Analysis. Cambridge Univ. Press, Cambrdige (1993)
9. Taylor, M.: Pseudodifferential Operators. Princeton Univ. Press, Princeton, NJ (1981)
10. Taylor, M.: Partial Differential Equations. Vols. 1-3. Springer-Verlag, New York (1996)
11. Taylor, M.: Eigenfunction expansions and the Pinsky phenomenon on compact manifolds. J. Fourier Anal. Appl. **7**, 507-522 (2001)
12. Taylor, M.: The Gibbs phenomenon, the Pinsky phenomenon, and variants for eigenfunction expansions. Commun. Parital Differ. Equ. **27**, 565-605 (2002)
13. Ziller, W.: Geometry of the Katok examples. Ergod. Theor. Dyn. Sys. **3**, 135-157 (1982)

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